Sound Probabilistic Numerical Error Analysis

How do we compute the distribution of numerical errors at the output?

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Programming with Numerical Errors

```
def func(x:Real, y:Real, z:Real): Real = {
   val res = -3.79*x - 5.44*y + 9.73*z + 4.52
   return res
}
```

- **Reals** are implemented in **Floating point**/**Fixed point** data type

Programming with Numerical Errors

```
        (x:Float32, y:Float32, z:Float32): Float32
def func(x:Real, y:Real, z:Real): Real = {
   val res = -3.79*x - 5.44*y + 9.73*z + 4.52
   return res
}
```

- **Reals** are implemented in **Floating point**/**Fixed point** data type
- Introduces **round-off error**

Programming with Numerical Errors

```
        (x:Float32, y:Float32, z:Float32): Float32
def func(x:Real, y:Real, z:Real): Real = {
    val res = -3.79*x - 5.44*y + 9.73*z + 4.52
    return res
}
```

We need to bound the round-off error

- **Reals** are implemented in **Floating point/Fixed point** data type
- Introduces **round-off error**

State-of-the-art: Worst Case Error Analysis

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {
require (0.0 <= x <= 4.6 && 0.0 <= y, z <= 10.0)
    val res = -3.79*x - 5.44*y + 9.73*z + 4.52
    return res
}
```

Daisy FLUCTUAT

Gappa rosa FPTaylor

....

State-of-the-art: Worst Case Error Analysis

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {
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   return res
}
```

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....

Worst case error: **0.002**

Computes **absolute** round-off error

State-of-the-art: Worst Case Error Analysis

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {
require (0.0 <= x <= 4.6 && 0.0 <= y, z <= 10.0)
   val res = -3.79*x - 5.44*y + 9.73*z + 4.52
   return res
}
```

Daisy FLUCTUAT

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....

Worst case error: **0.002**

Occurs only with probability 0.002 !

Error Resilient Applications

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {
require (0.0 <= x <= 4.6 && 0.0 <= y, z <= 10.0)
   val res = -3.79*x - 5.44*y + 9.73*z + 4.52
   return res
}
```

Applications may tolerate large infrequent errors

A controller system can tolerate big errors while stabilizing

Error Resilient Applications

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {
require (0.0 <= x <= 4.6 && 0.0 <= y, z <= 10.0)
   val res = -3.79*x - 5.44*y + 9.73*z + 4.52
   return res +/- error
} ensuring (error <= 0.00199, 0.85)
```

Application tolerates big errors occurring with **<=** **0.15** probability

Applications may tolerate large infrequent errors

Worst Case Analysis = poor resource utilization

```
        (x:Float64, y:Float64, z:Float64): Float64
def func(x:Float32, y:Float32, z:Float32): Float32 = {
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   val res = -3.79*x - 5.44*y + 9.73*z + 4.52
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} ensuring (error <= 0.00199, 0.85)
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Applications may tolerate large infrequent errors

With only **Worst case Analysis**, we need to change **precision**

Worst Case Analysis = poor resource utilization

```
            (x:Float64, y:Float64, z:Float64): Float64
def func(~~x:Float32, y:Float32, z:Float32): Float32~~ = {
require (0.0 <= x <= 4.6 && 0.0 <= y, z <= 10.0)
   val res = -3.79*x - 5.44*y + 9.73*z + 4.52
   return res +/- error
} ensuring (error <= 0.00199, 0.85)
```

Application tolerates big errors occurring with **<=** **0.15** probability

Applications may tolerate large infrequent errors

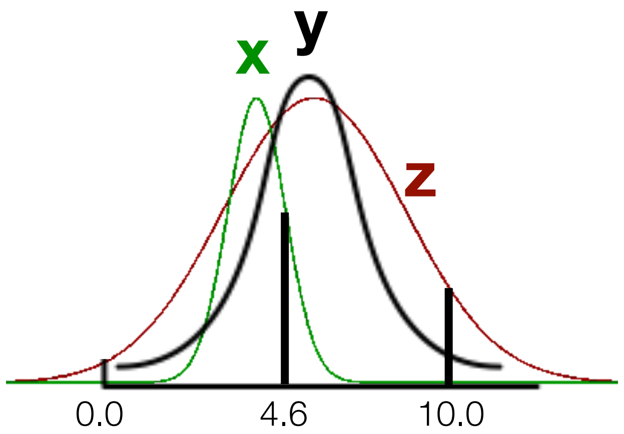
With only **Worst case Analysis**, we need to change **precision**

Need to consider the **probability distributions** of **inputs**

Our Goal: Probabilistic Analysis

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {

    x:= gaussian(0.0, 4.6)
    y:= gaussian(0.0, 10.0)
    z:= gaussian(0.0, 10.0)
```

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We consider **probability distributions** of **inputs**

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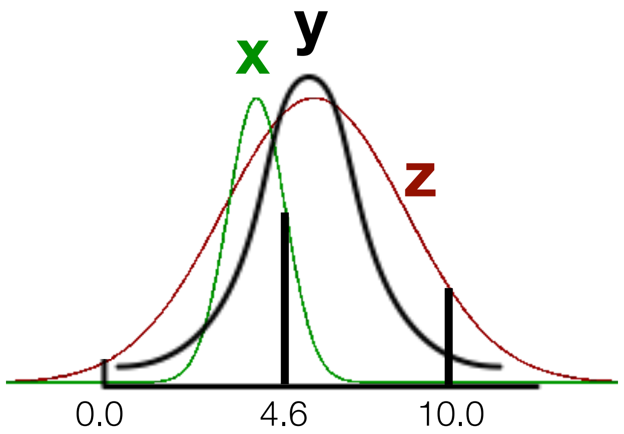

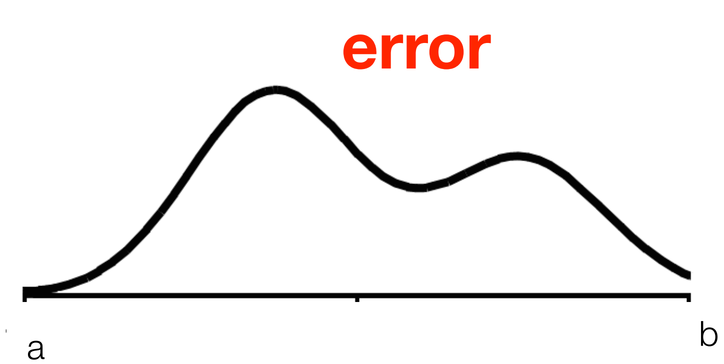

- Compute **probability distribution** of **error**

Our Goal: Probabilistic Analysis

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {

    x:= gaussian(0.0, 4.6)
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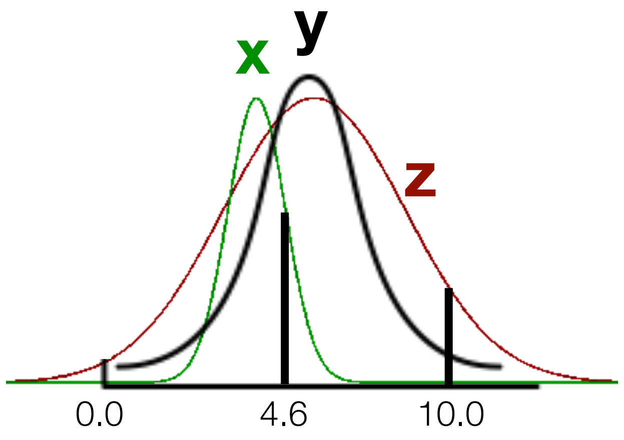

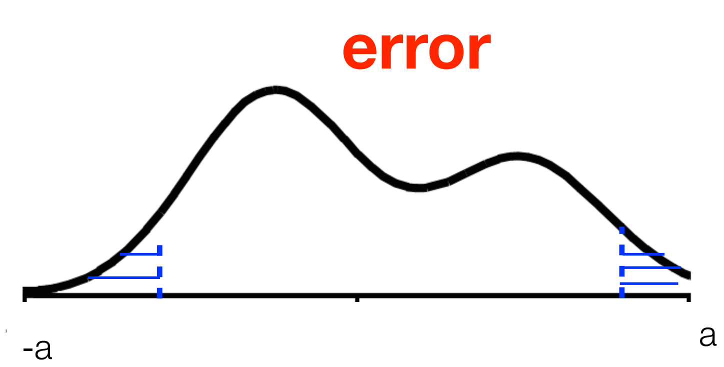

- Compute **probability distribution** of **error**
- Compute a **smaller error** given a **threshold**

Approximate Hardware

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {

    x:= gaussian(0.0, 4.6)
    y:= gaussian(0.0, 10.0)
    z:= gaussian(0.0, 10.0)

    val res = -3.79*x - 5.44*y + 9.73*z + 4.52
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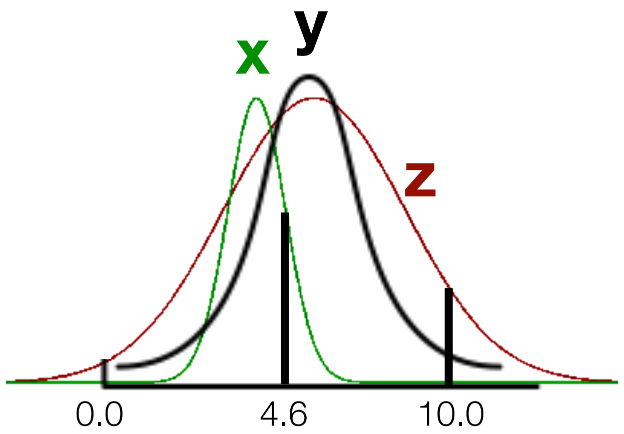

What happens if we have **Approximate Hardware** with **Probabilistic Error Specification**?

Probabilistic Analysis for Approximate Hardware

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {

    x:= gaussian(0.0, 4.6)
    y:= gaussian(0.0, 10.0)
    z:= gaussian(0.0, 10.0)

    val res = -3.79*x - 5.44*y + 9.73*z + 4.52
    return res +/- error
} ensuring (error <= 0.00199, 0.85)
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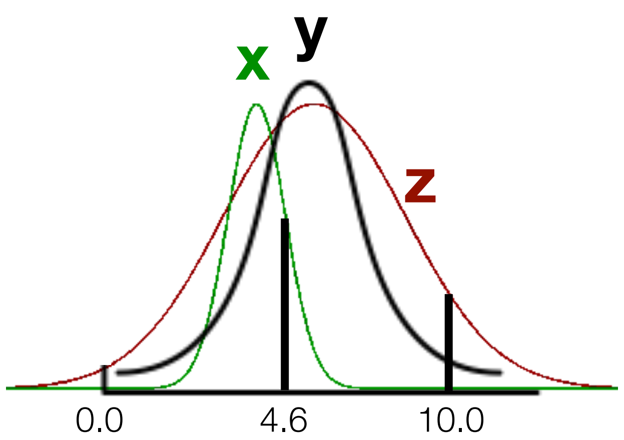


Error Specification: **<0.00199, 0.9>, <0.00499, 0.1>**

Can we compute a **smaller** error given 0.85 as **threshold**?

Contributions

- Sound analysis of probabilistic numerical errors
  - considers probability distribution of inputs and computes error distribution

- Application on Approximate Hardware
  - considers probability distribution of error specification

- Prototype implementation on top of Daisy
  https://github.com/malyzajko/daisy/tree/probabilistic

Overview: Sound Analysis

Finite Precision Program with Probabilistic Inputs

```
def func(..) {
 x := gaussian(0.0, 4.6)
 y := gaussian(0.0, 10.0)
 z := gaussian(0.0, 10.0)
 res = -3.79*x - 5.44*y + 9.73*z + 4.52
 return res
}
```

Probabilistic Round-off Error Analysis

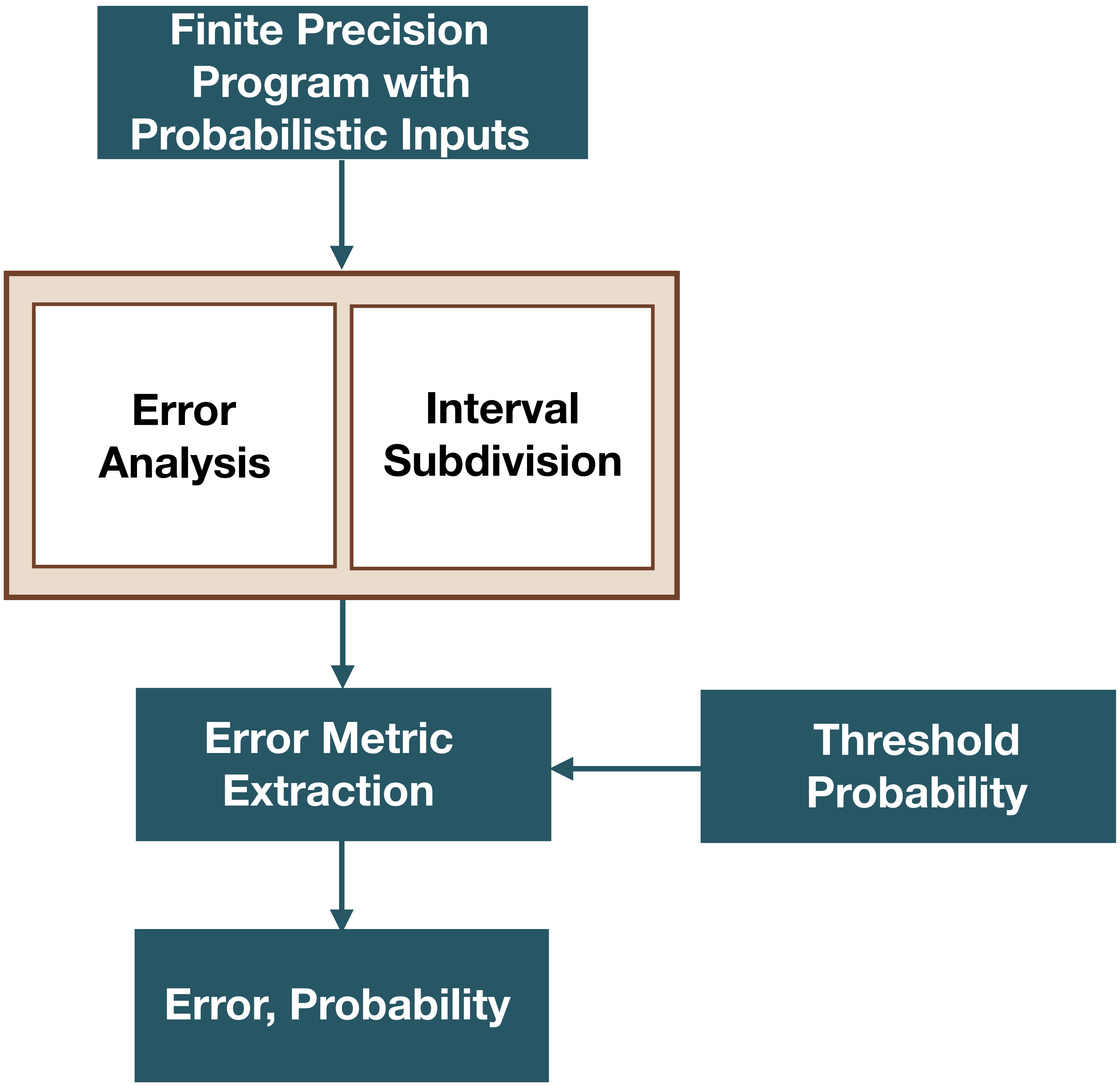
Error Metric Extraction

Threshold Probability

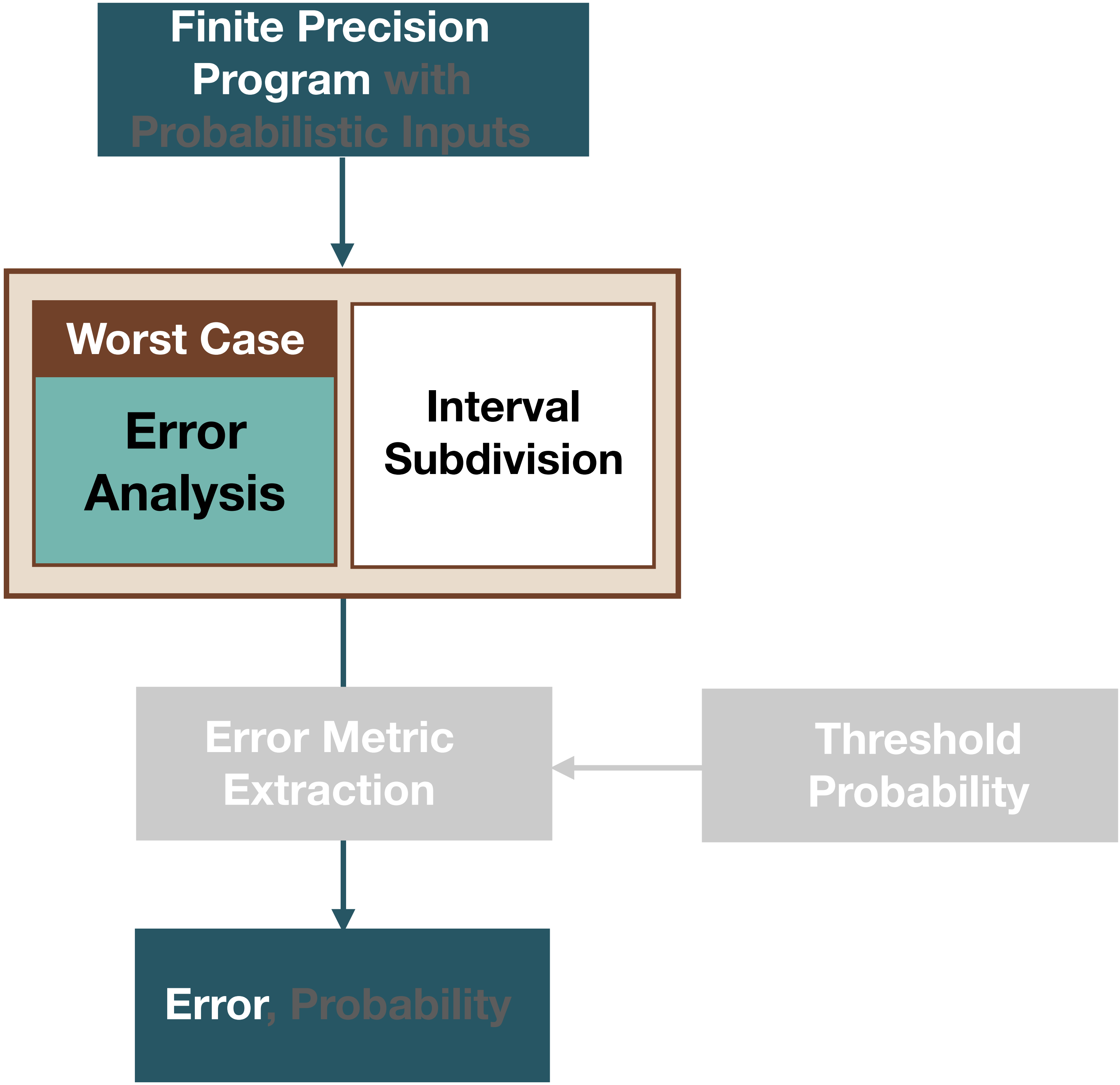
0.85

Error, Probability

Overview: Sound Analysis



Before going into Probabilistic Analysis...



Background: Worst Case Error Analysis

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {
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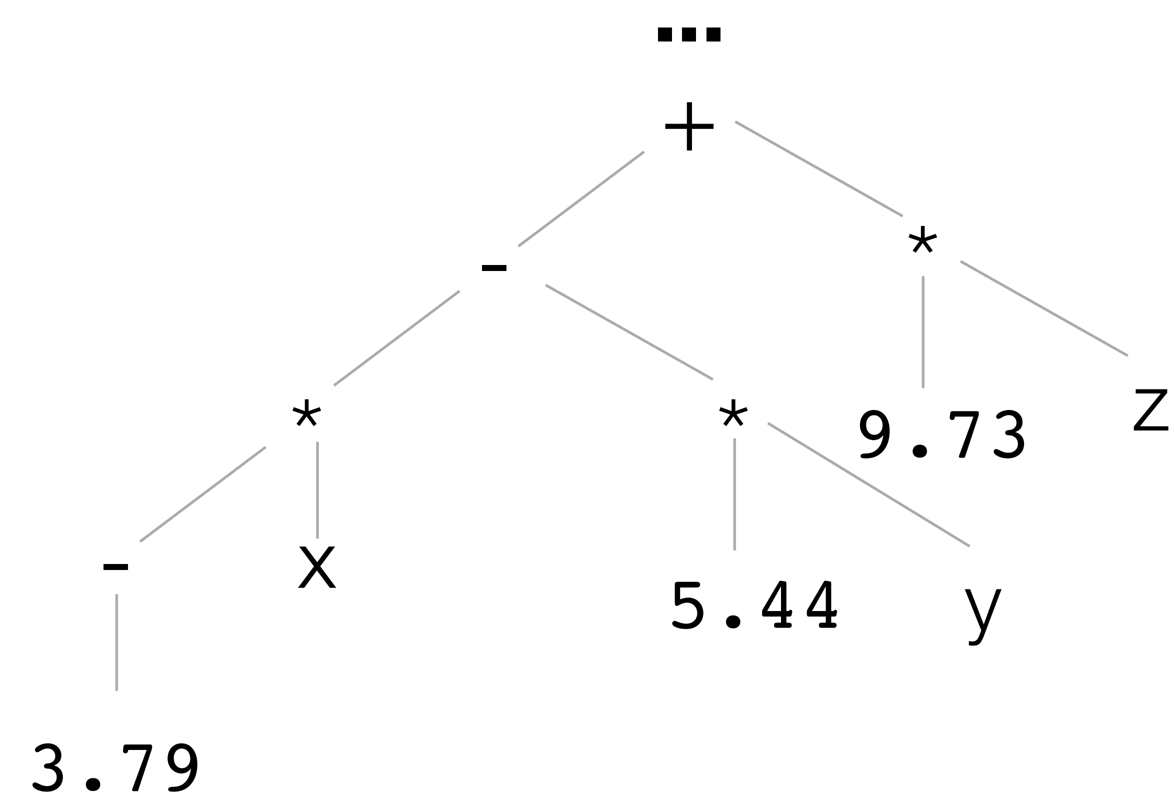

Background: Worst Case Error Analysis

```
def func(x:Float32, y:Float32, z:Float32): Float32 = {
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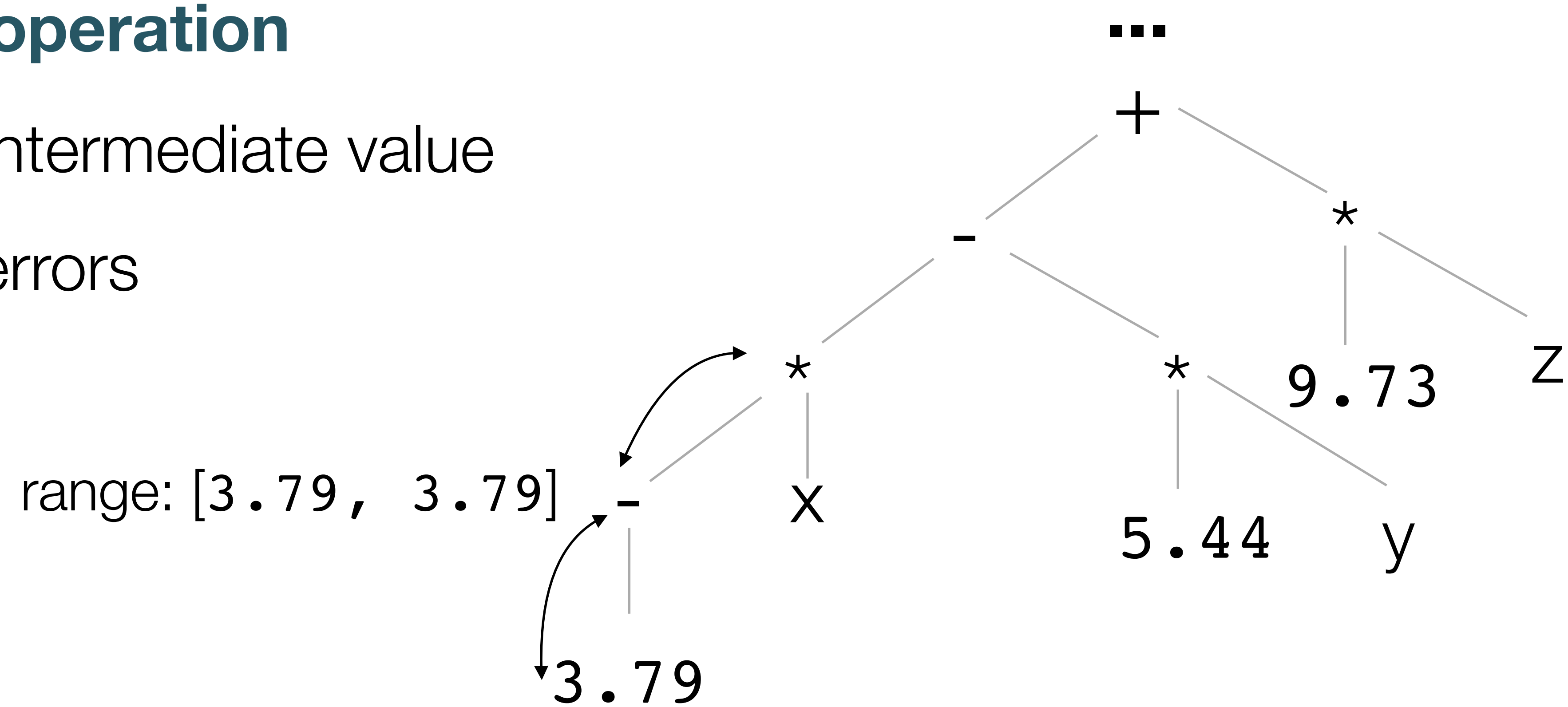
```

For **each arithmetic operation**

- compute range for intermediate value
- propagate existing errors



Uses **Interval** / **Affine Arithmetic**

Background: Worst Case Error Analysis

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def func(x:Float32, y:Float32, z:Float32): Float32 = {
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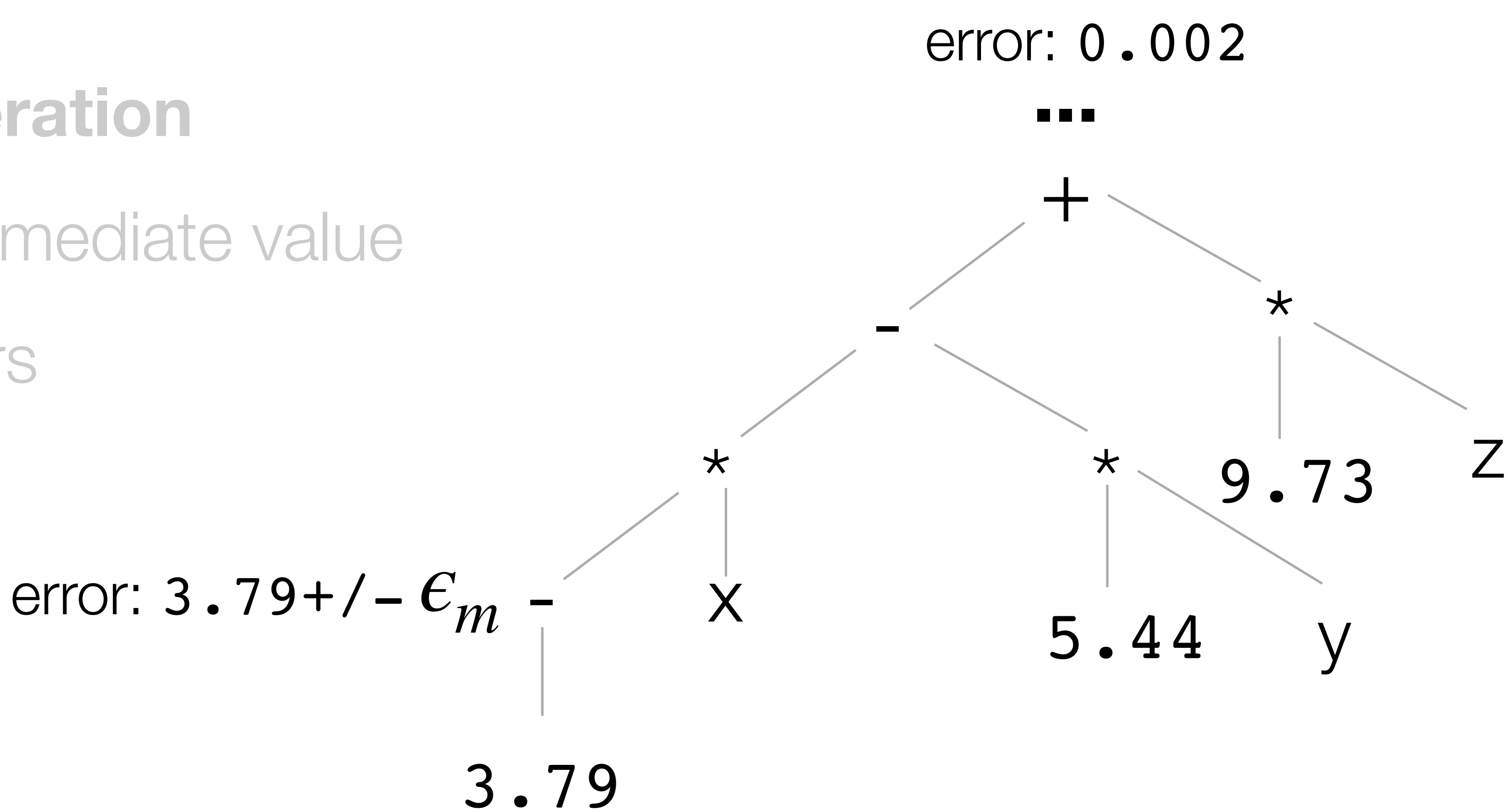

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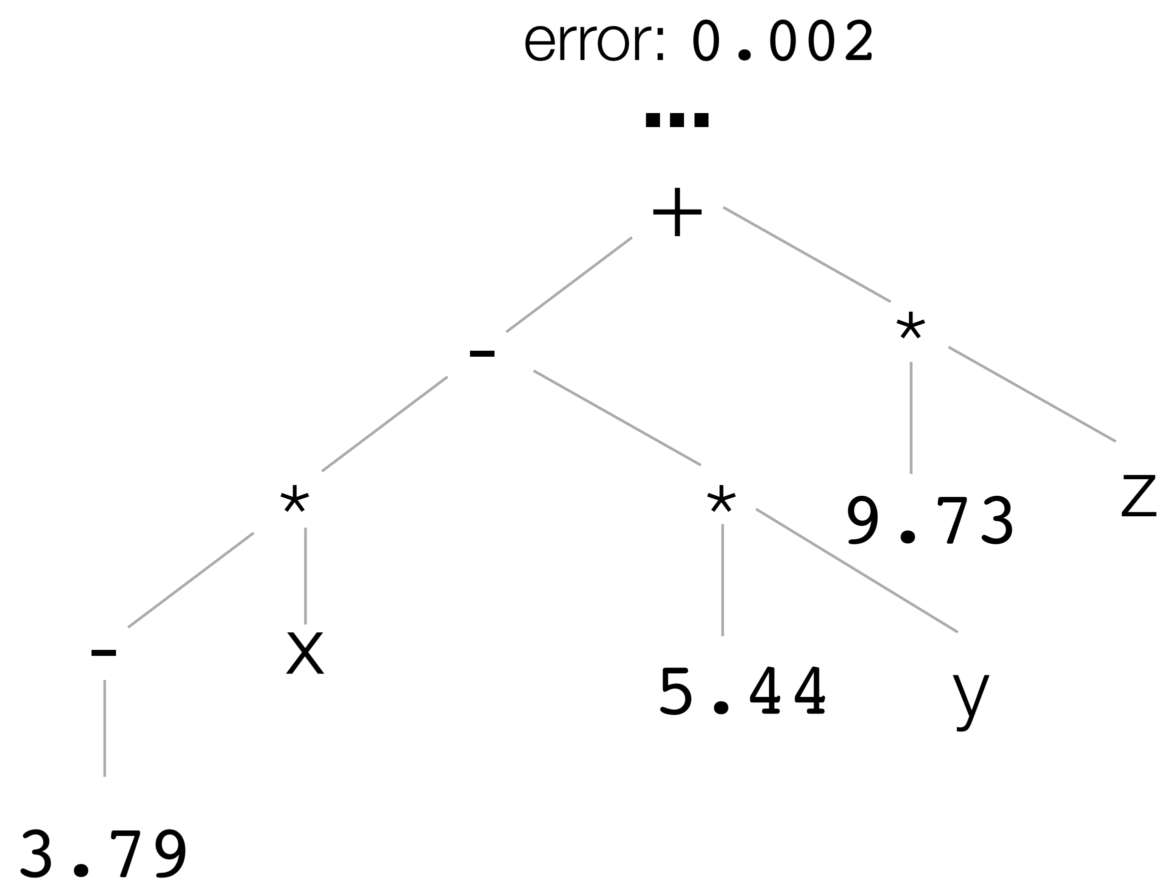
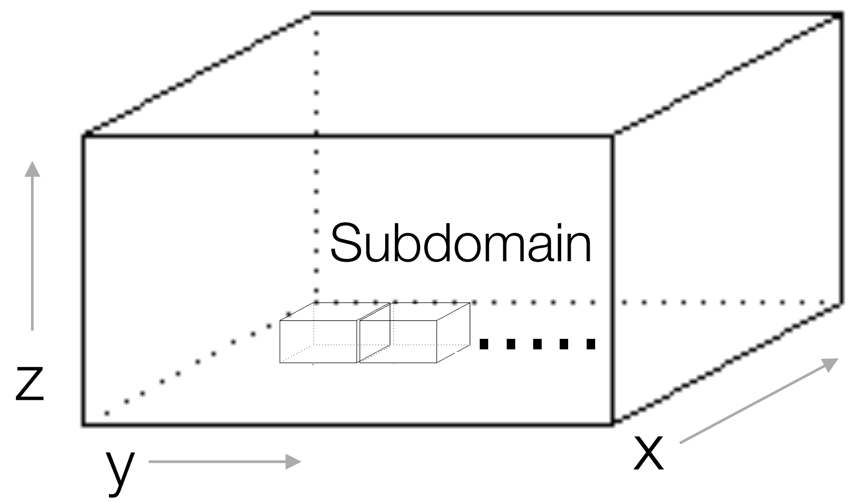


To compute **precise** error, **subdivide** the intervals

Background: Interval Subdivision

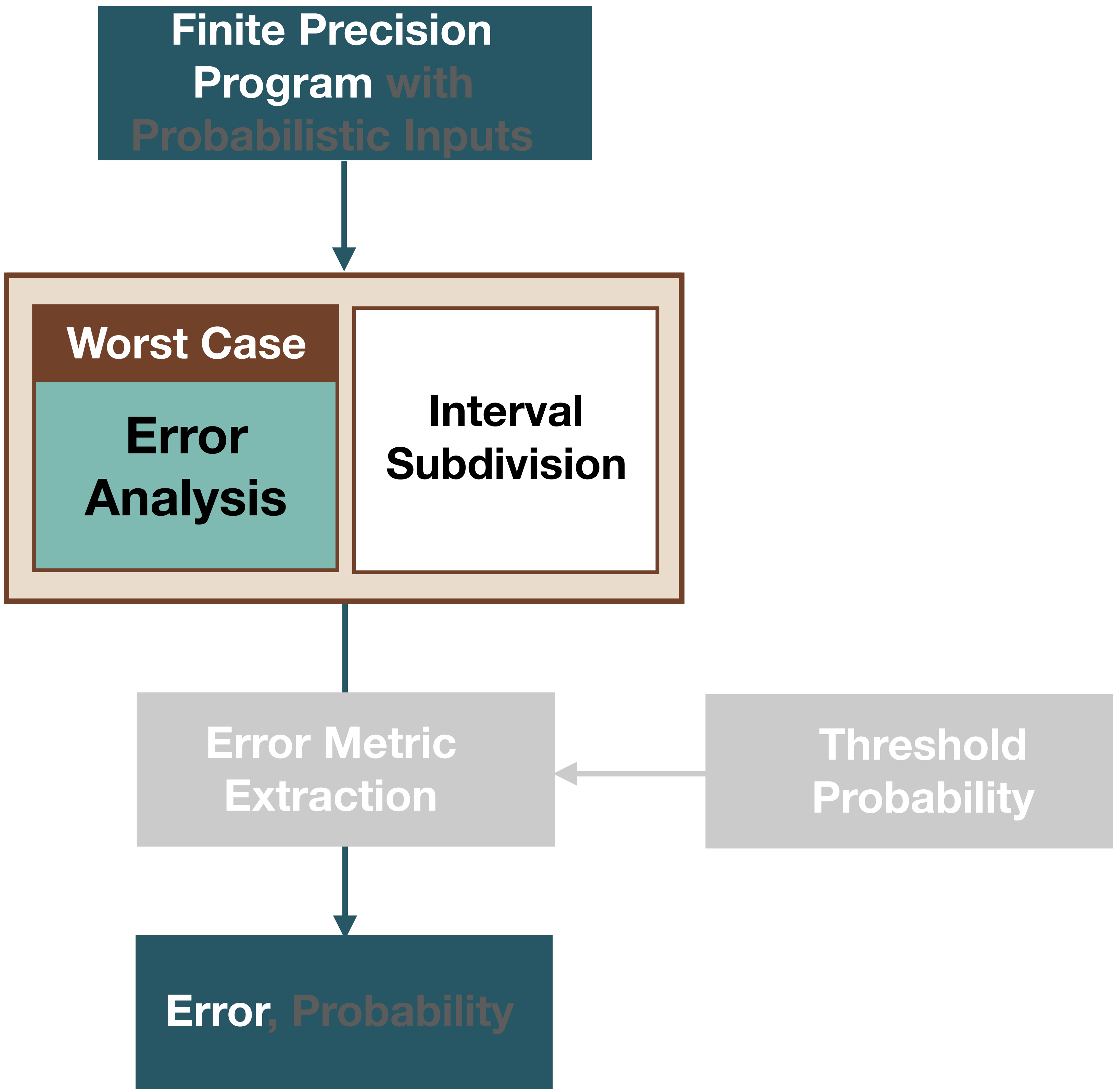
Input Space:

`0.0 <= x <= 4.6 && 0.0 <= y, z <= 10.0`

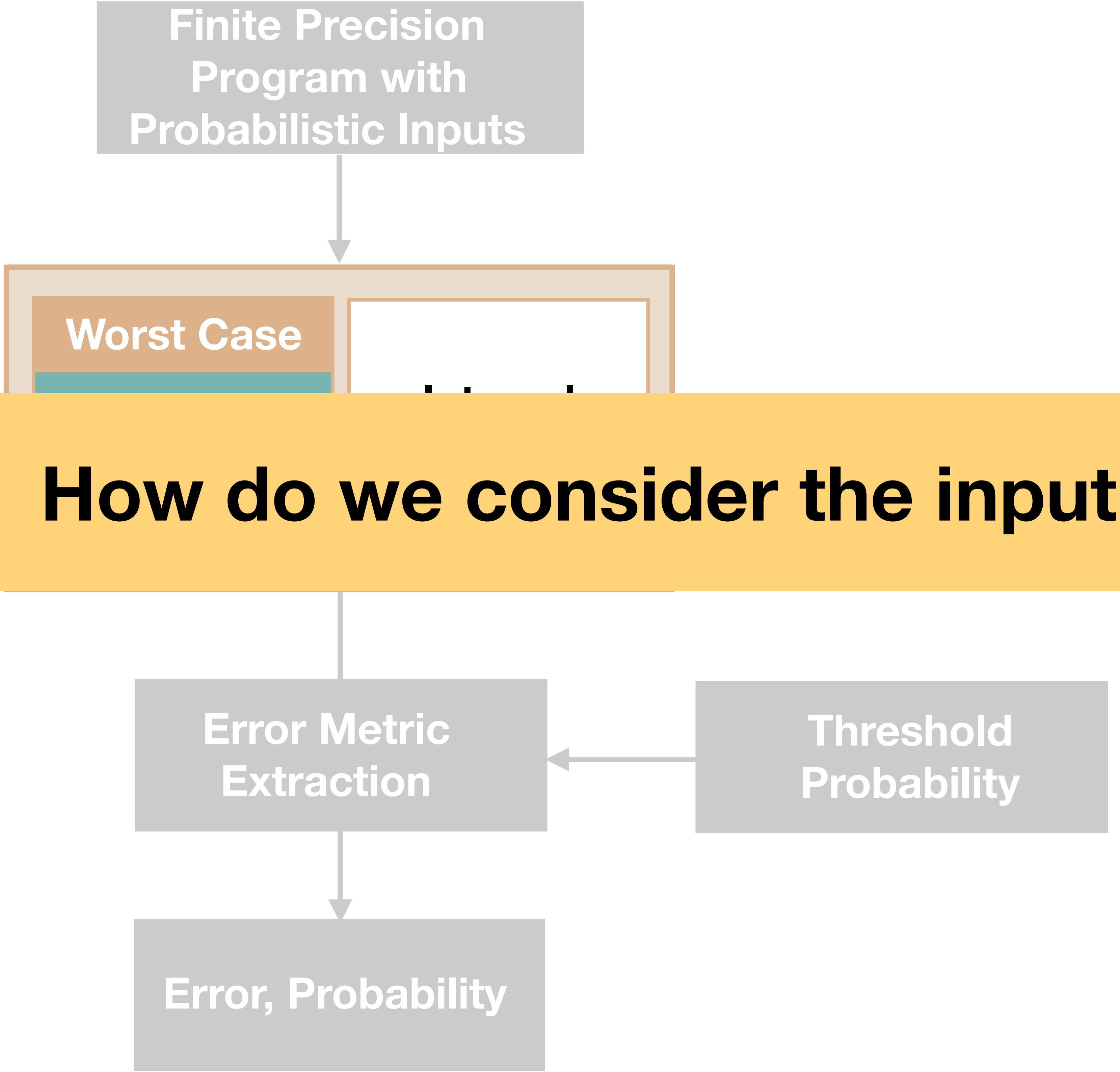


- Generate subdomains
- For each subdomain, compute the **worst case error**
- Return the max abs error

Background: Worst Case Analysis with Subdivision

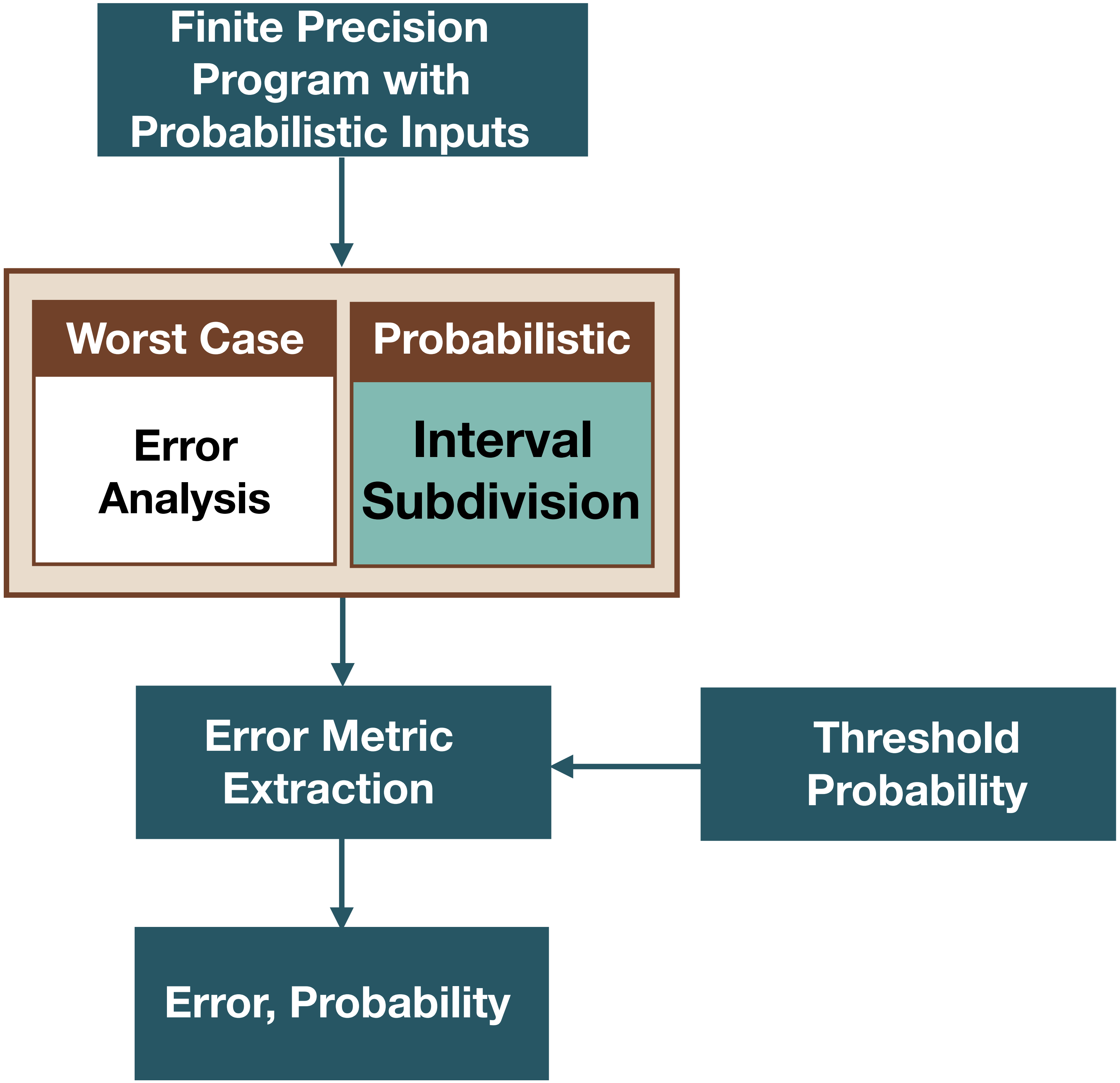


Background: Worst Case Analysis with Subdivision



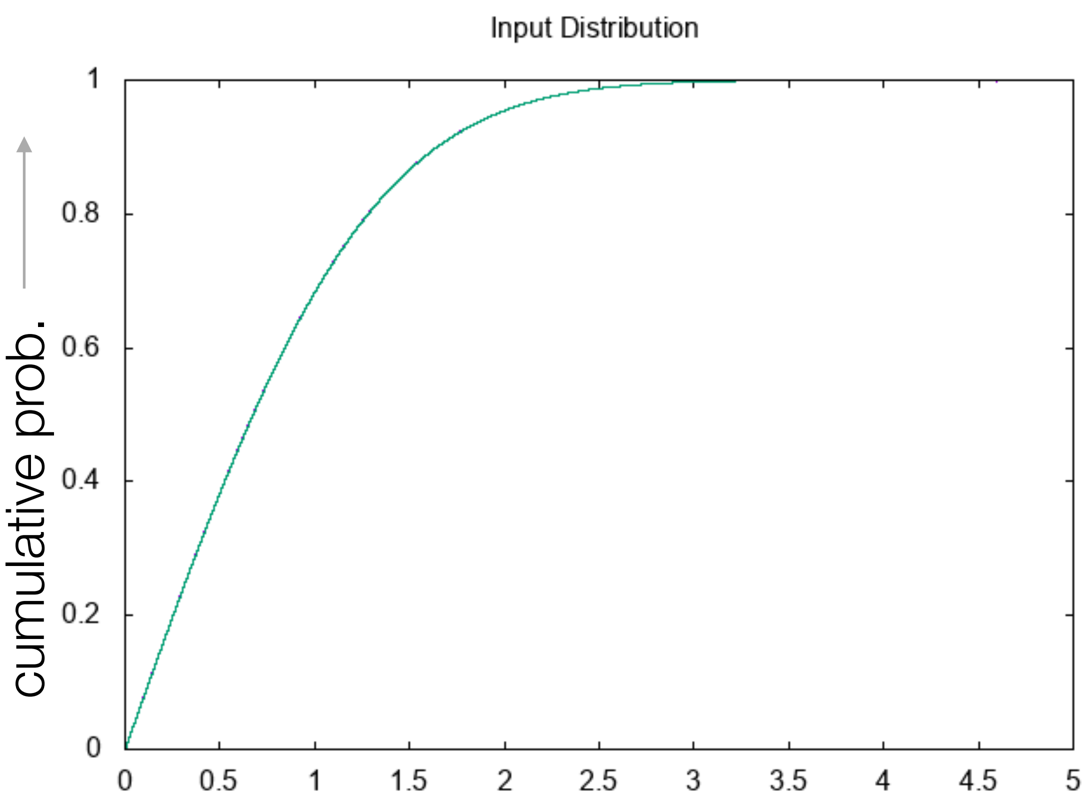
How do we consider the input distributions?

Overview: Sound Probabilistic Analysis



Probabilistic Interval Subdivision

```
x := gaussian(0.0, 4.6)
```




```
def func(..) {
 x := gaussian(0.0, 4.6)
 y := gaussian(0.0, 10.0)
 z := gaussian(0.0, 10.0)
 res = -3.79*x - 5.44*y + 9.73*z + 4.52
 return res
}
```

Probabilistic Interval Subdivision

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x := gaussian(0.0, 4.6)
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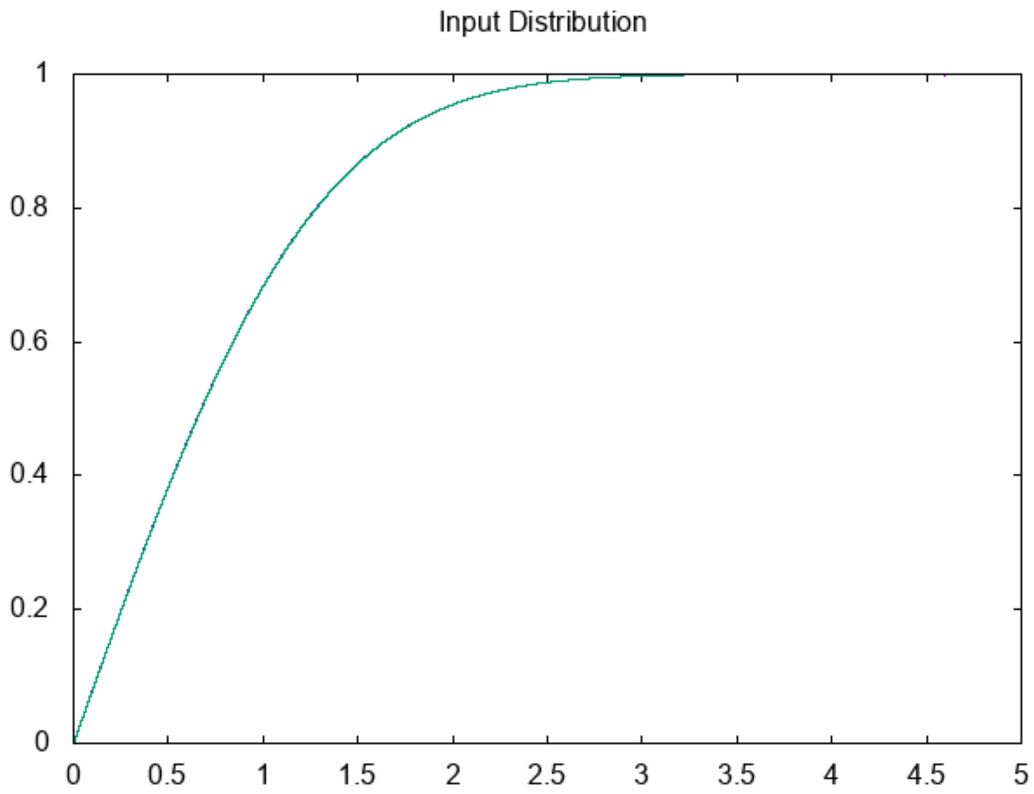
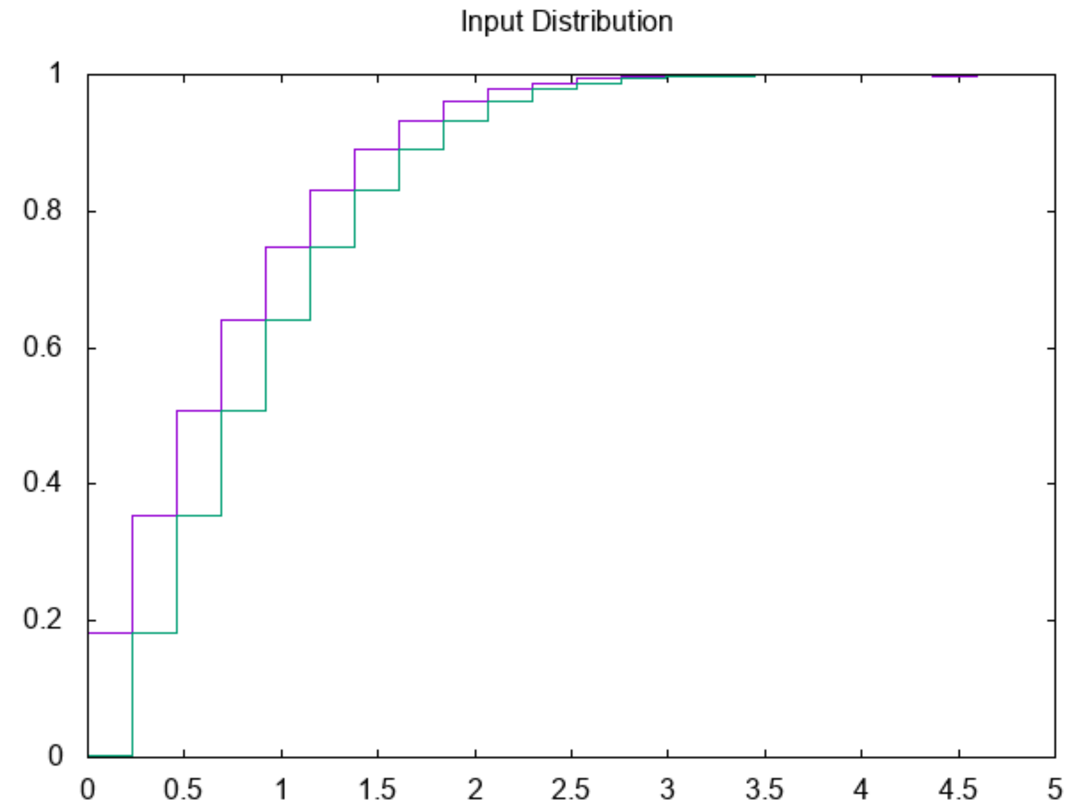


Discretize

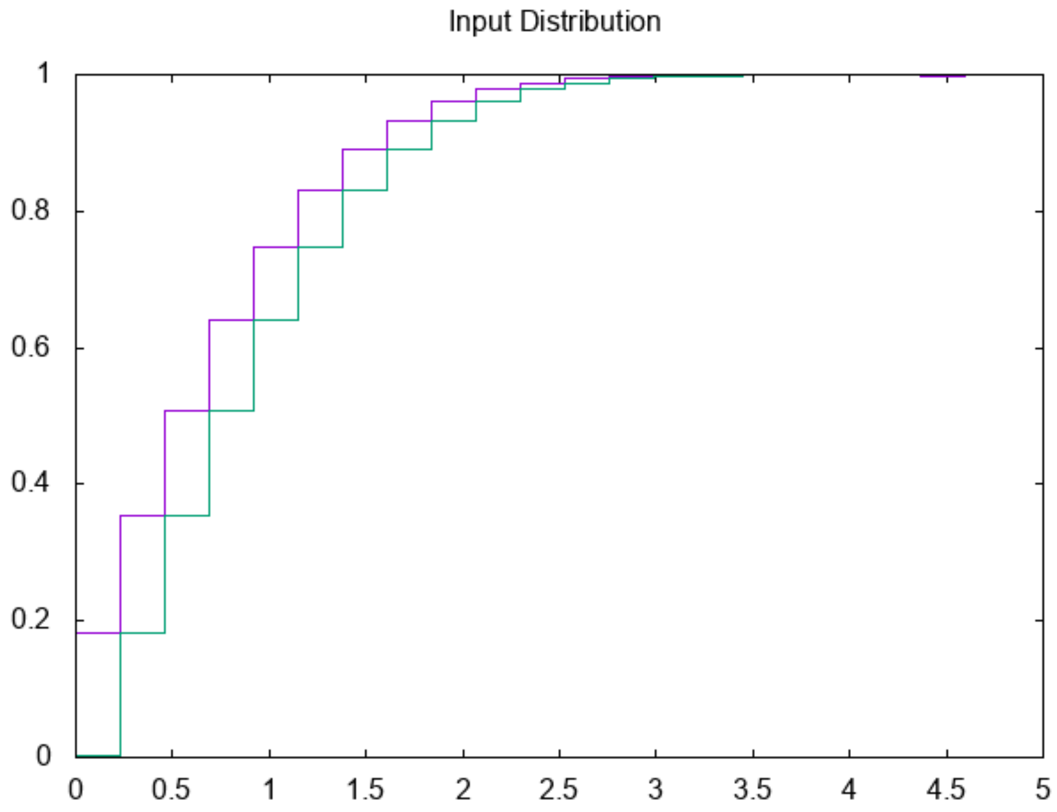


Discretize the continuous distribution into a **set** of **intervals** and **probabilities**

Probabilistic Interval Subdivision

```
x := gaussian(0.0, 4.6)
```

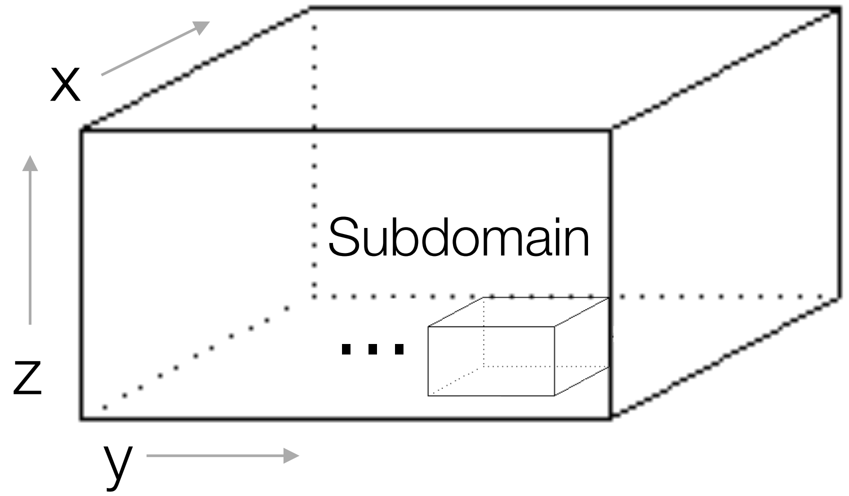
```
<[2.30, 4.60], 0.02>
          .
          .
          .
<[0.20, 0.45], 0.17>
<[0.00, 0.20], 0.18>
```



Discretize the continuous distribution into a **set** of **intervals** and **probabilities**

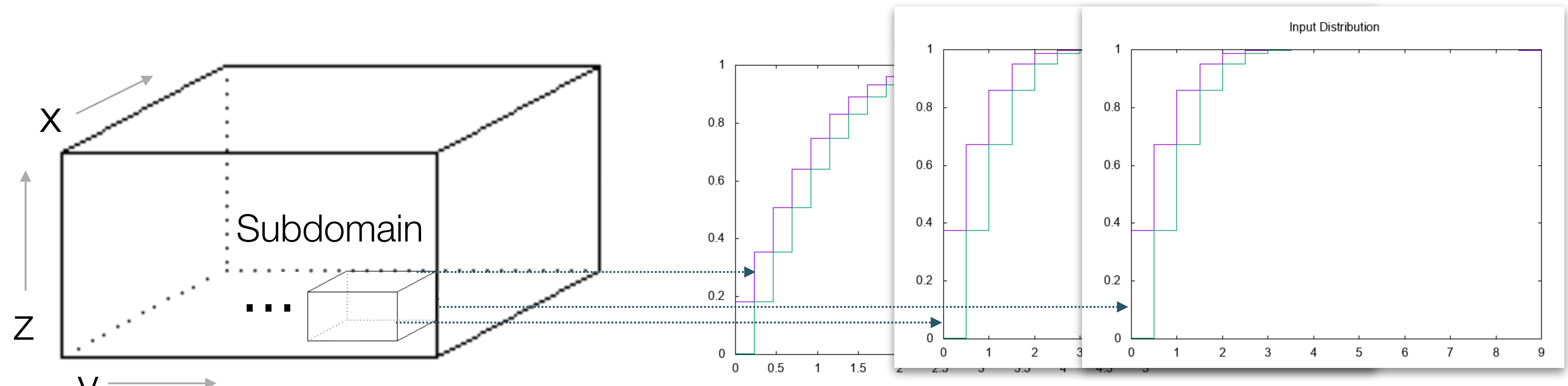
Probabilistic Interval Subdivision

Input Space:



Probabilistic Interval Subdivision

Input Space:



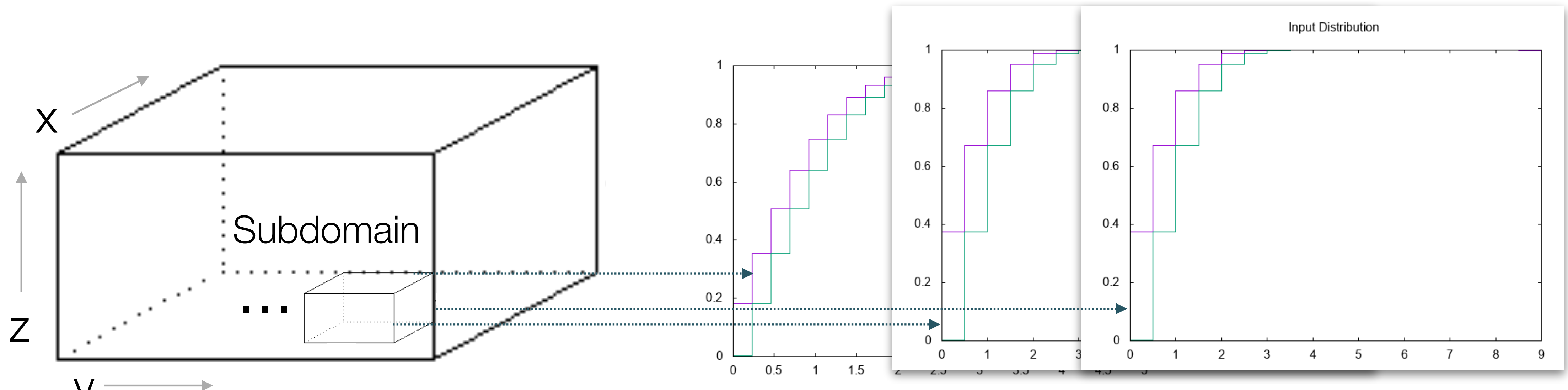
`x:=gaussian(0.0, 4.6)` `y:=gaussian(0.0, 10.0)` `z:=gaussian(0.0, 10.0)`

- A set of subdomains with probabilities by taking **Cartesian product**

$$\forall i \in x, \forall j \in y, \forall k \in z, s_{ijk} = x_i \times y_j \times z_k$$

Probabilistic Interval Subdivision

Input Space:



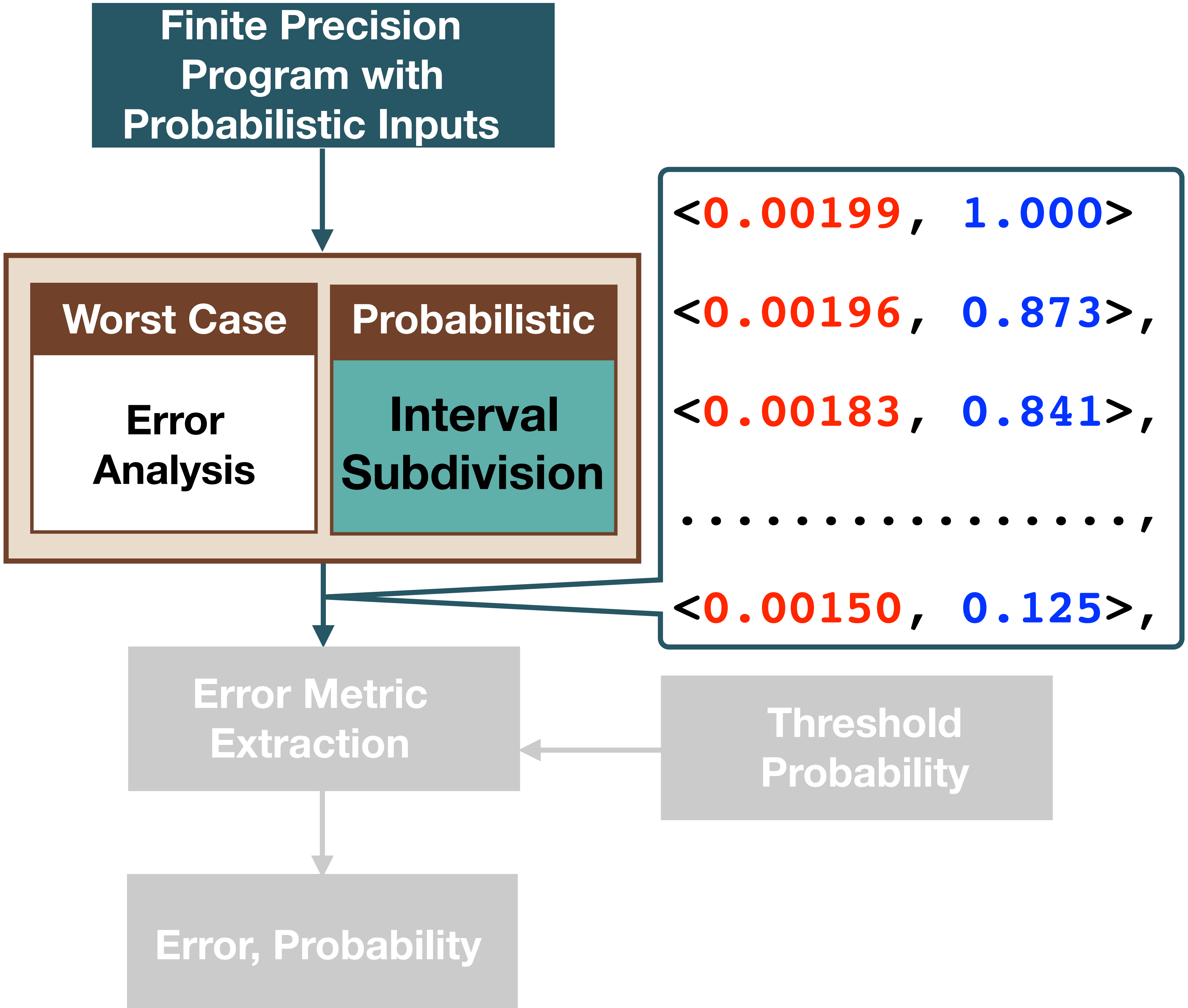
`x:=gaussian(0.0, 4.6)` `y:=gaussian(0.0, 10.0)` `z:=gaussian(0.0, 10.0)`

- A set of subdomains with probabilities by taking **Cartesian product**

$$\forall i \in x, \forall j \in y, \forall k \in z, s_{ijk} = x_i \times y_j \times z_k$$

- **Worst Case Error Analysis** for each subdomain

Computed errors and probabilities



Overview: Sound Analysis

Finite Precision Program with Probabilistic Inputs

Worst Case — Error Analysis

Probabilistic — Interval Subdivision

Error Metric Extraction

Threshold Probability

0.85

Error, Probability

largest error

<0.00199, 1.000>

<0.00196, 0.873>,

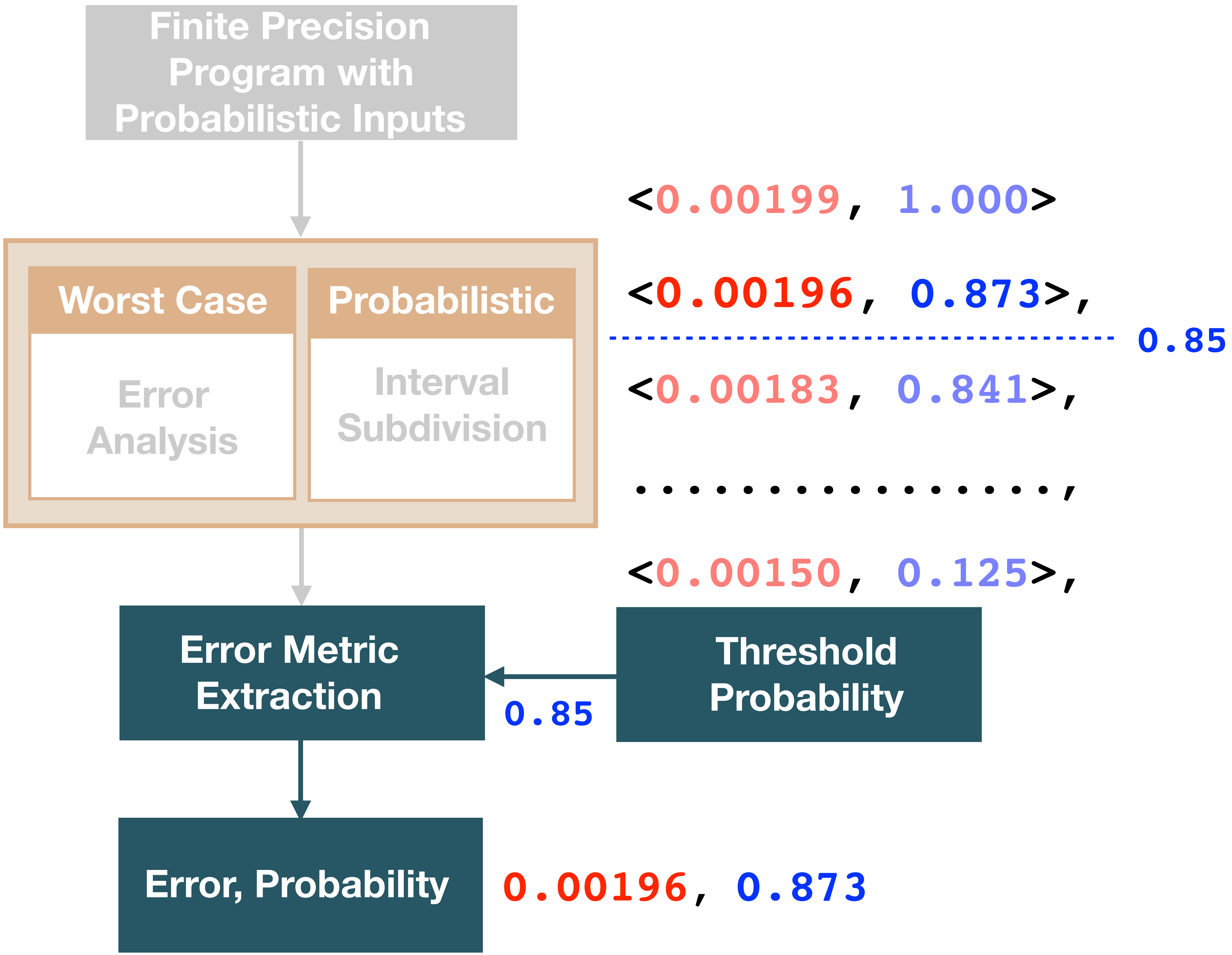
0.85

<0.00183, 0.841>,

.................,

<0.00150, 0.125>,

Overview: Sound Analysis



Results: Probabilistic Interval Subdivision

Benchmarks	**Worst Case** (state-of-the-art)
sineOrder3	4.62E-07
sqrt	1.50E-04
bspline1	2.09E-07
rigidbody2	1.94E-02
traincar2	1.37E-03
filter4	6.51E-06
cubic	1.83E-05
classIDX0	8.77E-06
polyIDX1	6.81E-04
neuron	3.22E-05

Worst case errors for 32 bit floating-point and gaussian input distributions

Results: Probabilistic Interval Subdivision

Benchmarks	**Worst Case** (state-of-the-art)	**Prob. Subdivision**
sineOrder3	4.62E-07	**2.97E-07**
sqrt	1.50E-04	**8.38E-05**
bspline1	2.09E-07	**1.96E-07**
rigidbody2	1.94E-02	**1.06E-02**
traincar2	1.37E-03	**1.32E-03**
filter4	6.51E-06	**6.09E-06**
cubic	1.83E-05	**1.73E-05**
classIDX0	8.77E-06	**7.95E-06**
polyIDX1	6.81E-04	**4.51E-04**
neuron	3.22E-05	**3.20E-05**

Reduction w.r.t. the worst case with 0.85 threshold probability for 32 bit floating-point and gaussian input distributions

Reduction using Probabilistic Interval Subdivision

Benchmarks	Worst Case (state-of-the-art)	Prob. Subdivision (% reduction)
sineOrder3	4.62E-07	**-35.7**
sqrt	1.50E-04	**-44.1**
bspline1	2.09E-07	**-6.2**
rigidbody2	1.94E-02	**-45.4**
traincar2	1.37E-03	**-3.6**
filter4	6.51E-06	**-6.5**
cubic	1.83E-05	**-5.5**
classIDX0	8.77E-06	**-9.4**
polyIDX1	6.81E-04	**-33.8**
neuron	3.22E-05	**-0.6**

Reduction w.r.t. the worst case with 0.85 threshold probability for 32 bit floating-point and gaussian input distributions

Reduction using Probabilistic Interval Subdivision

Benchmarks	Worst Case (state-of-the-art)	Prob. Subdivision (% reduction)
sineOrder3	4.62E-07	**-35.7**
sqrt	1.50E-04	**-44.1**
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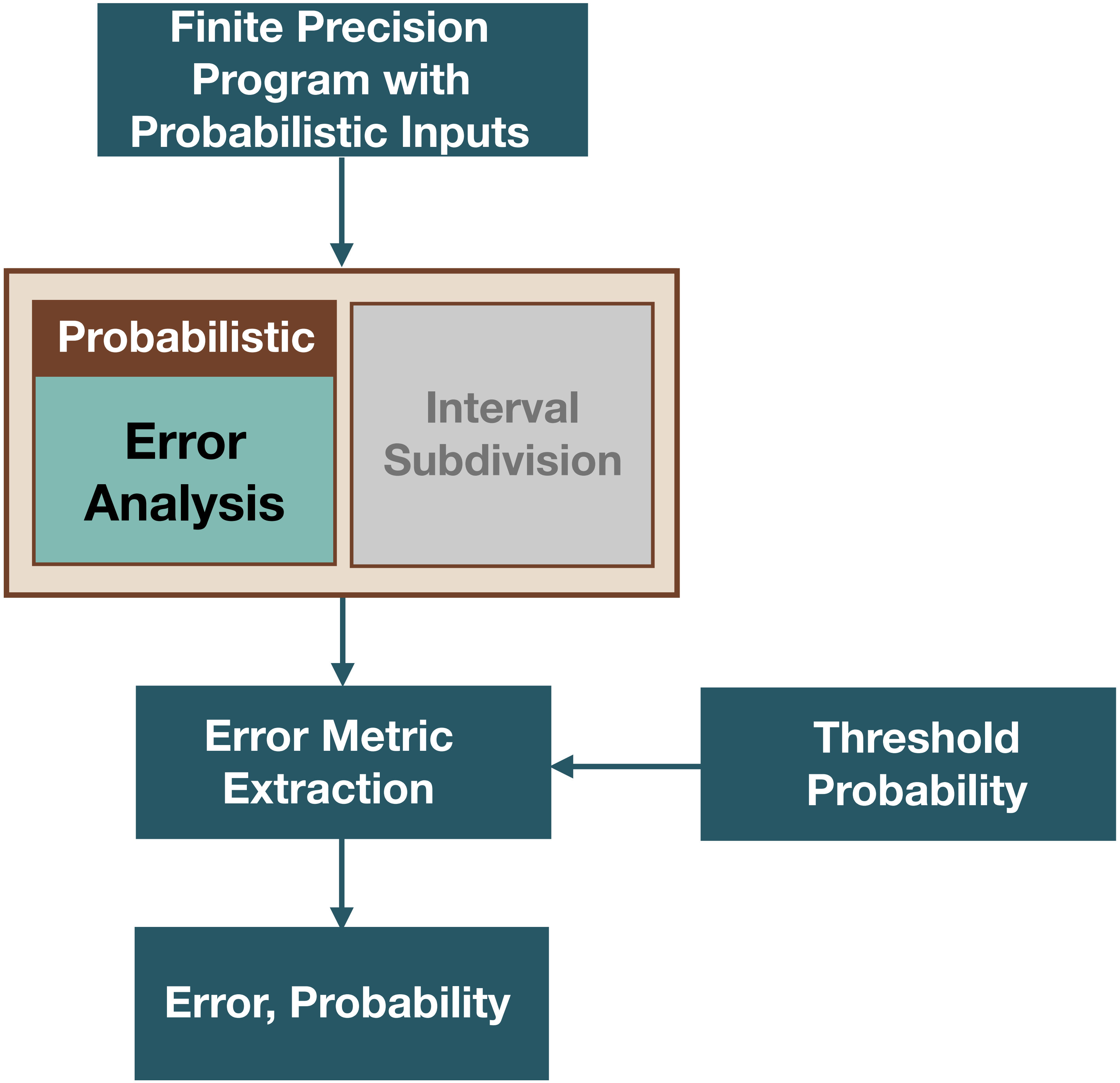
cubic	1.83E-05	**-5.5**
classIDX0	8.77E-06	**-9.4**
polyIDX1	6.81E-04	**-33.8**
neuron	3.22E-05	**-0.6**

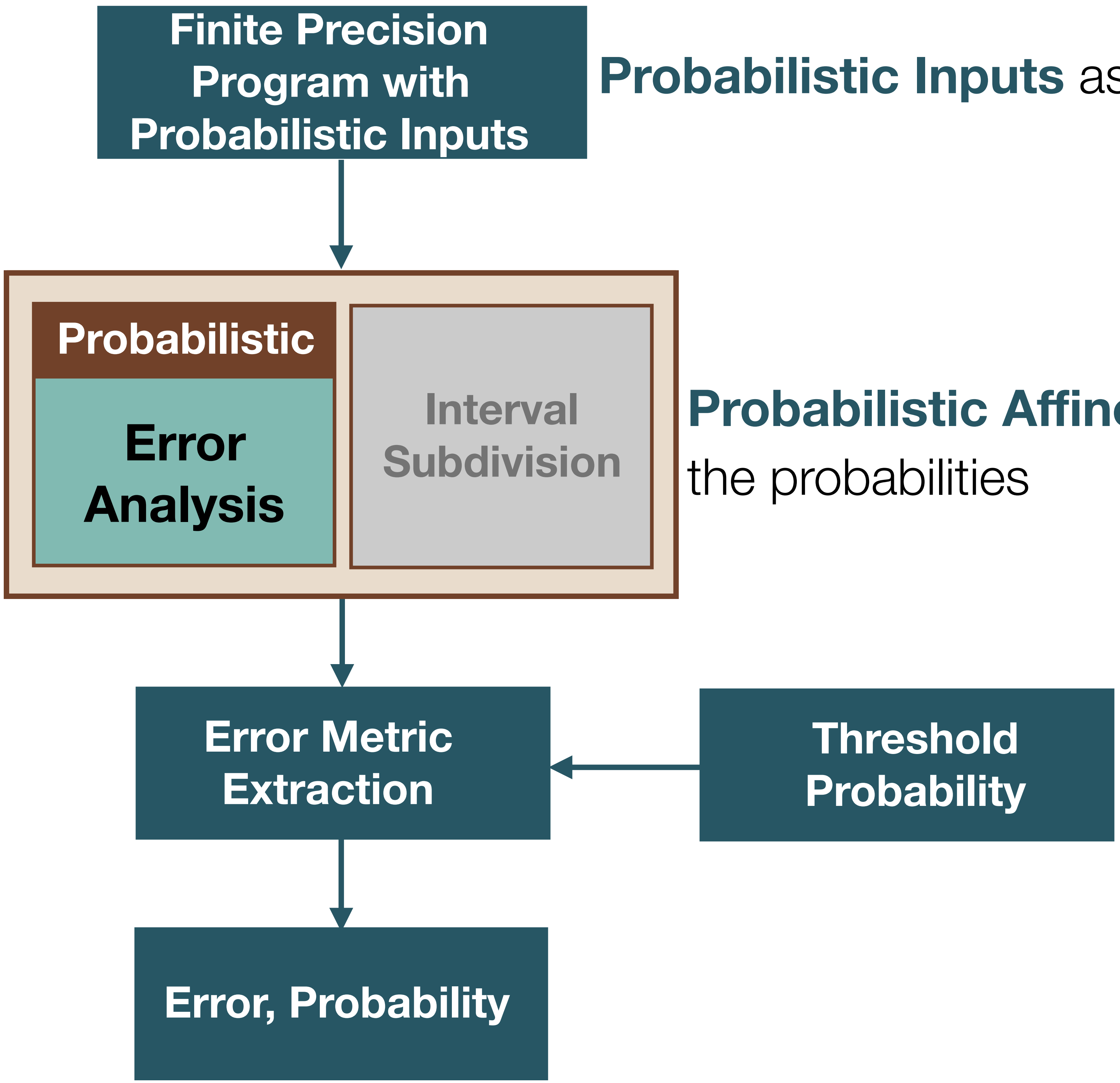
Reduction w.r.t. the worst case with 0.85 threshold probability for 32 bit floating-point and gaussian input distributions

Still computes the worst case error!

Overview: Sound Probabilistic Error Analysis



Overview: Sound Probabilistic Error Analysis



Probabilistic Inputs as uncertain probabilities

Probabilistic Affine Arithmetic propagates the probabilities

Background: Probabilistic Affine Arithmetic

- Affine Arithmetic propagates linear relations between variables
- Dependencies are tracked using shared noise symbol

$$\hat{x} := x_0 + \sum_{i=1}^{p} x_i \epsilon_i, \ \epsilon_i \in [-1,1]$$

Noise Symbol

"Static Analysis of Programs with Imprecise Probabilistic Inputs", A. Adje, O. Bouissou, J. Goubault-Larrecq, E. Goubault, S. Putot, VSTTE 2013

Background: Probabilistic Affine Arithmetic

- Affine Arithmetic propagates linear relations between variables
- Dependencies are tracked using shared noise symbol
- Keeps the probabilities while tracking dependencies

$$< [a_1, b_1], w_1 > , \cdots < [a_n, b_n], w_n >$$

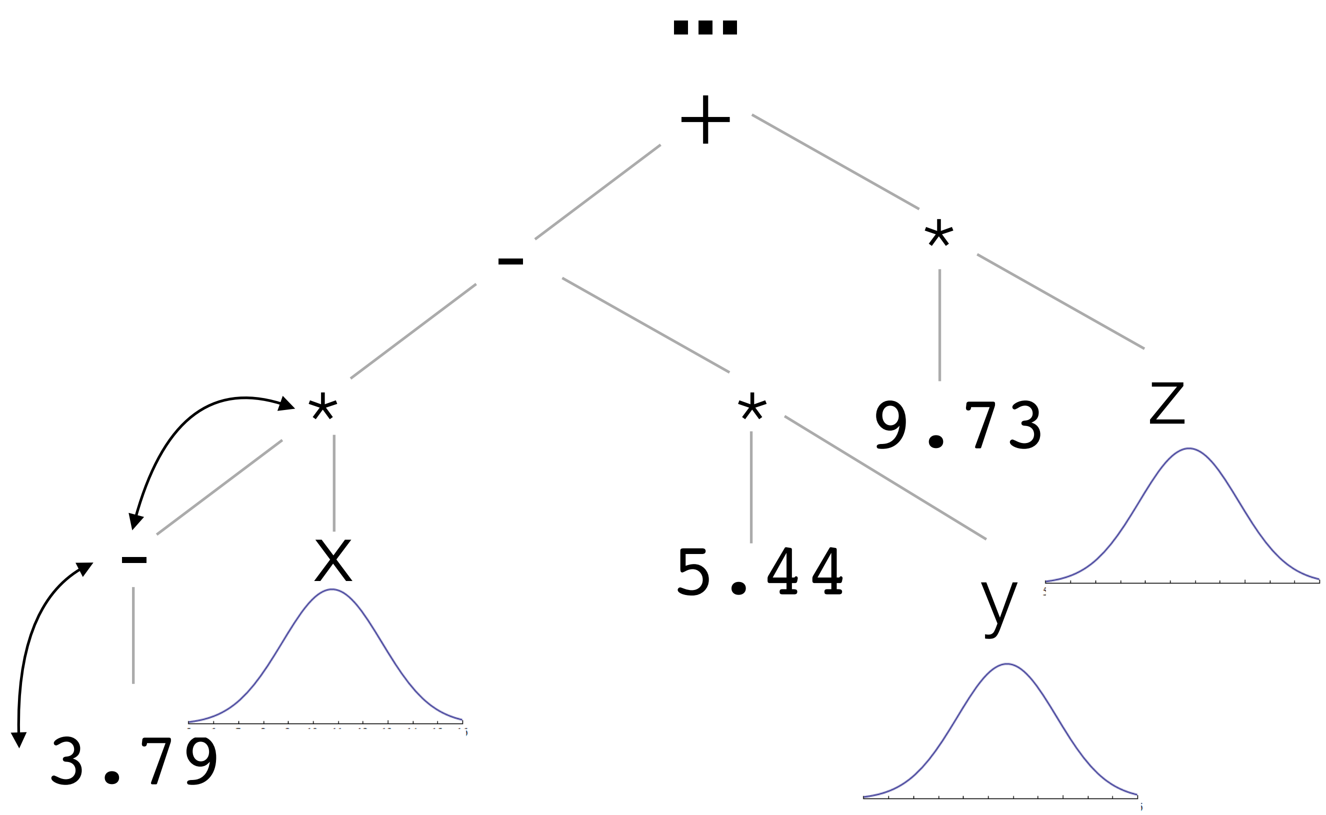
$$\hat{x} := x_0 + \sum_{i=1}^{p} x_i \epsilon_i,\ \epsilon_i \in [-1,1]$$

Noise Symbol

- Arithmetic operations are computed term wise

"Static Analysis of Programs with Imprecise Probabilistic Inputs", A. Adje, O. Bouissou, J. Goubault-Larrecq, E. Goubault, S. Putot, VSTTE 2013

Probabilistic Range Computation

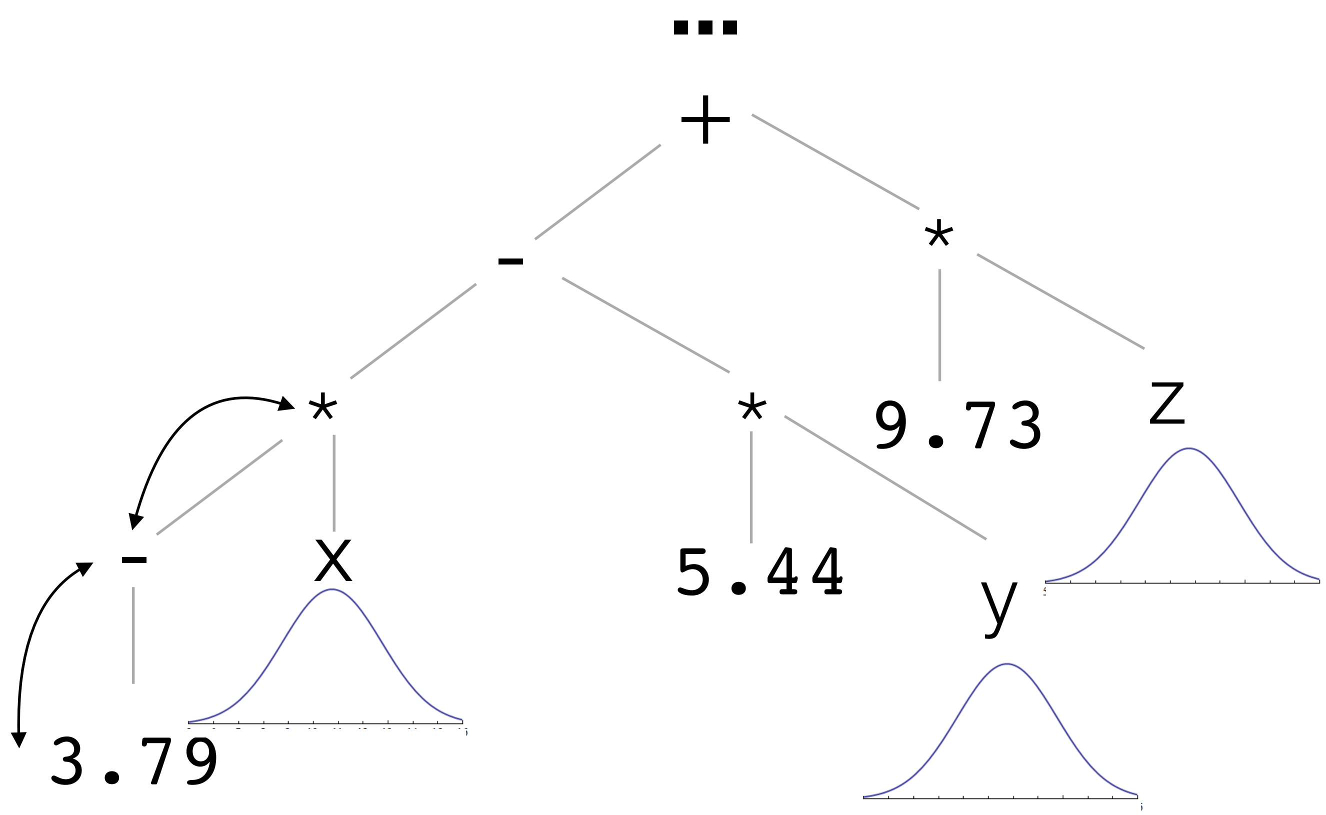


For **each arithmetic operation**

- compute range for intermediate value starting with initial distributions

"Discrete Choice in the Presence of Numerical Uncertainties", D. Lohar, E. Darulova, S. Putot, E. Goubault, EMSOFT 2018

Our Contribution: Probabilistic Error Computation



For **each arithmetic operation**

- compute range for intermediate value starting with initial distributions
- propagate existing errors
  - uses probabilistic affine arithmetic
- compute new errors
  - errors are added as fresh noise terms

Computed Error Distribution

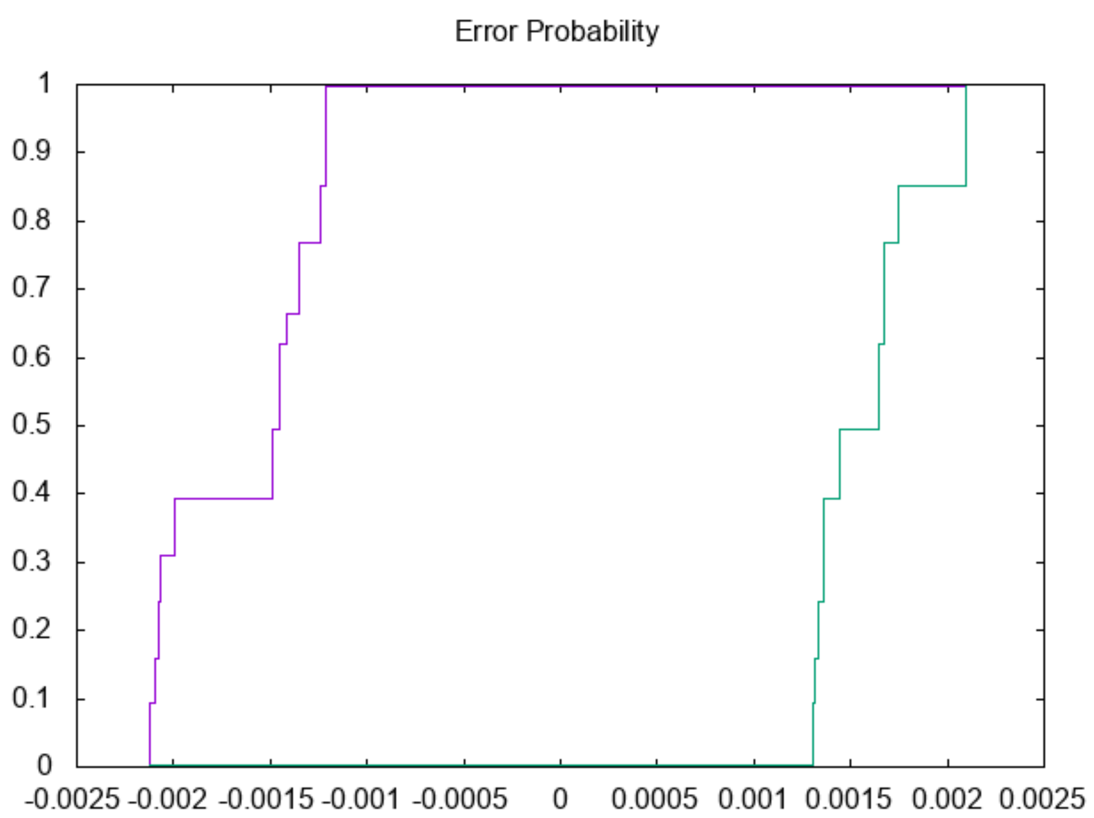
<[-0.0022, -0.0018], 1.0>,

<[-0.0018, 0.0012], 0.96>,

....

....

<[-0.0011, 0.0021], 0.14>



Generate a **set** of error **intervals** and probabilities

Computed Error Distribution

```
<[-0.0022, -0.0018], 1.0>,

<[-0.0018, 0.0012], 0.96>,
---------------------------------------------------------
.........................        Threshold probability = 0.85

.........................

<[-0.0011, 0.0021], 0.14>
```

- Repeat: remove <error, probability> if the total probability >= threshold

Computed Error Distribution

```
<[-0.0022, -0.0018], 1.0>,

<[abs(-0.0018), abs(0.0012)], 0.96>,
------------------------------------------------------------
........................

........................

<[abs(-0.0011), abs(0.0021)], 0.14>
```

Threshold probability = **0.85**

Error, Probability: **0.0021, 0.96**

- Repeat: remove <error, probability> if the total probability >= threshold
- Return the maximum error with probability

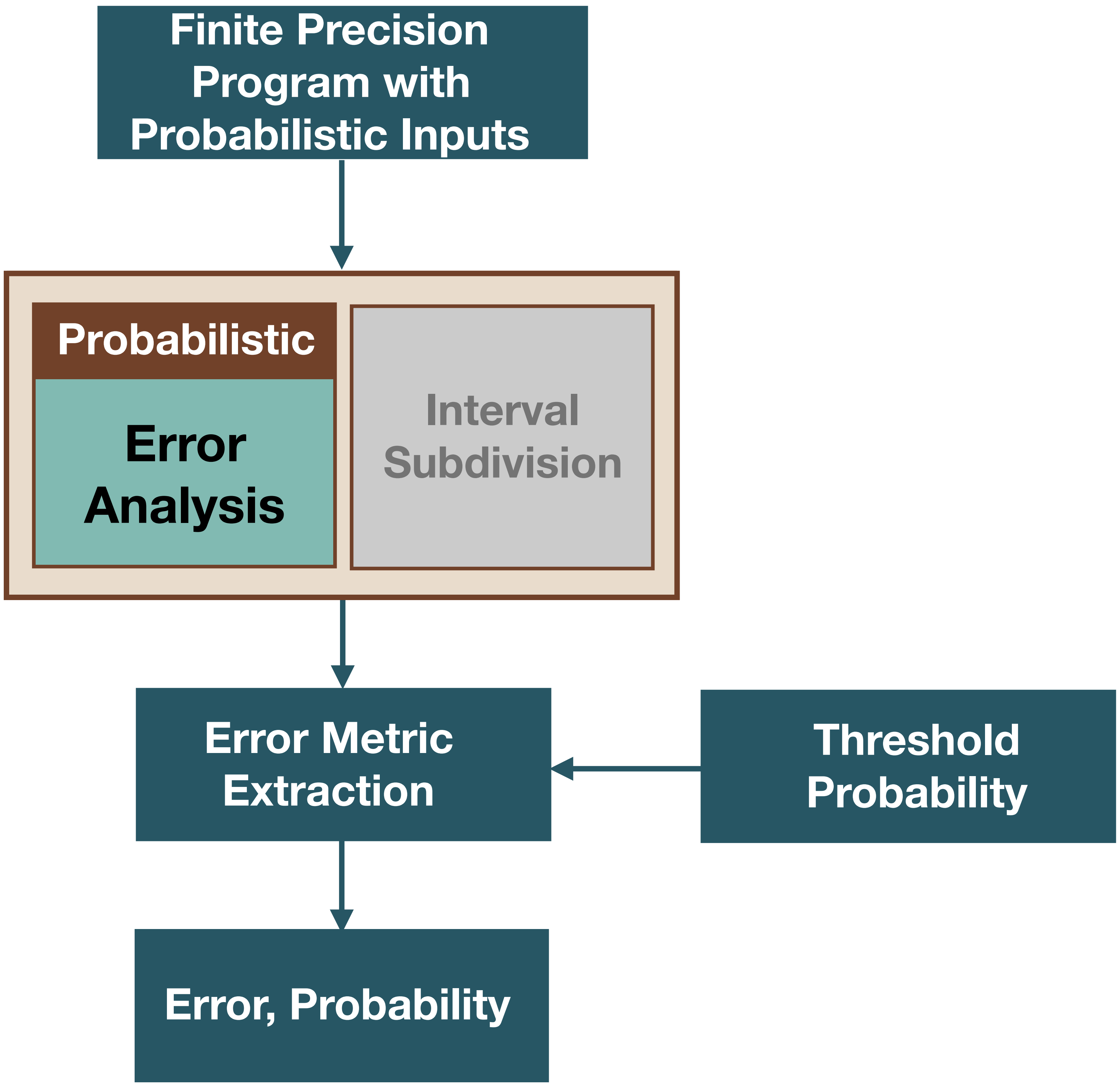
Results: Probabilistic Error Analysis

Benchmarks	Worst Case (state-of-the-art)	Prob. Subdivision (% Reduction)	Prob. Error (% Reduction)
sineOrder3	4.62E-07	-35.7	**42.2**
sqrt	1.50E-04	-44.1	**10.6**
bspline1	2.09E-07	-6.2	**84.2**
rigidbody2	1.94E-02	-45.4	**-50.0**
traincar2	1.37E-03	-3.6	**4.4**
filter4	6.51E-06	-6.5	**11.9**
cubic	1.83E-05	-5.5	**12.6**
classIDX0	8.77E-06	-9.4	**8.0**
polyIDX1	6.81E-04	-33.8	**3.9**
neuron	3.22E-05	-0.6	**>100**

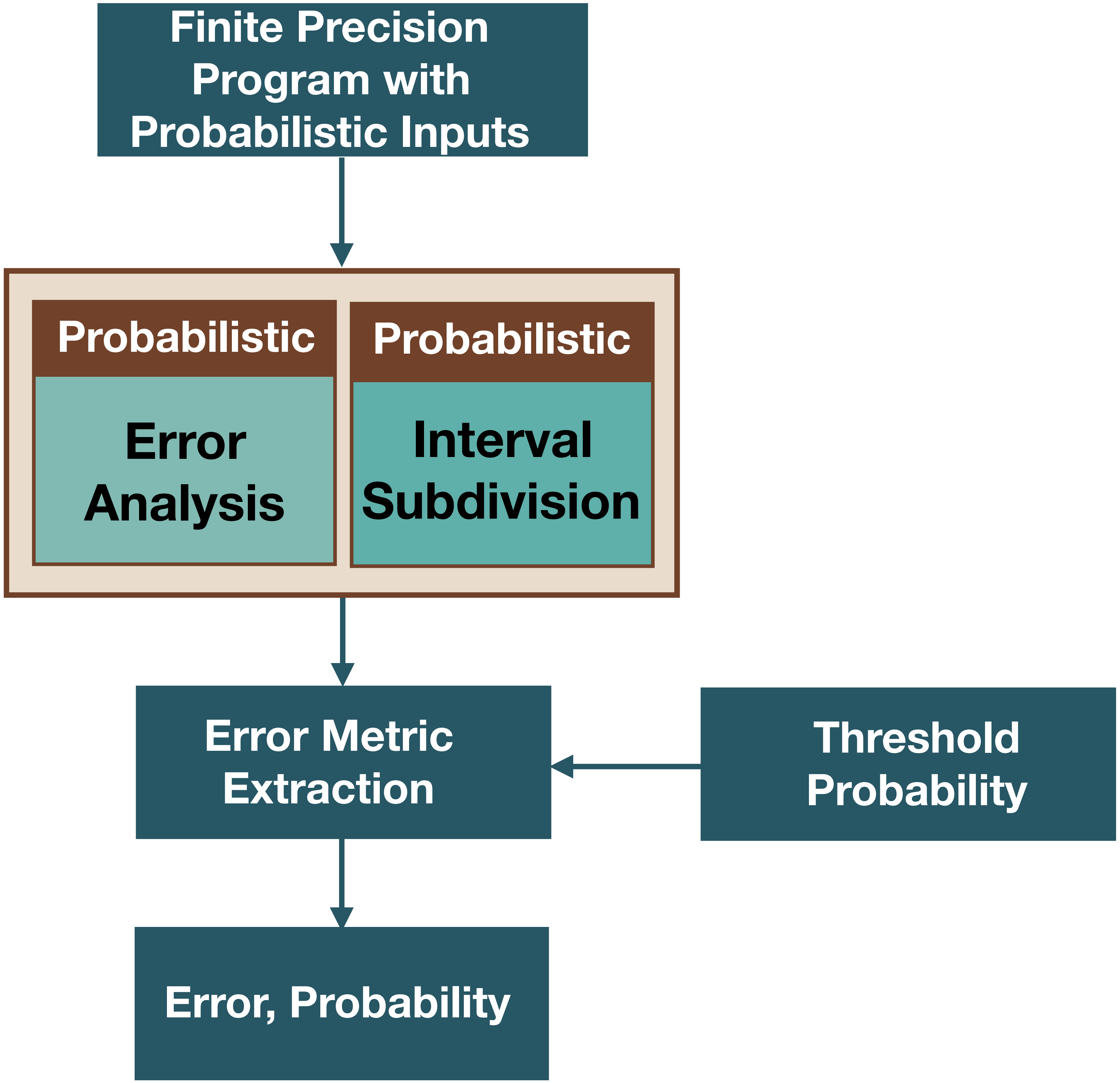
Reduction % with 0.85 threshold probability for 32 bit floating-point and gaussian input distributions

High over-approximation!

Overview: Sound Probabilistic Error Analysis



Overview: Sound Probabilistic Error Analysis



Overview: Sound Probabilistic Error Analysis

Finite Precision Program with Probabilistic Inputs

Probabilistic Error Analysis

Probabilistic Interval Subdivision

For each subdivision

- generate error distribution
- normalize with the probability of the subdomain

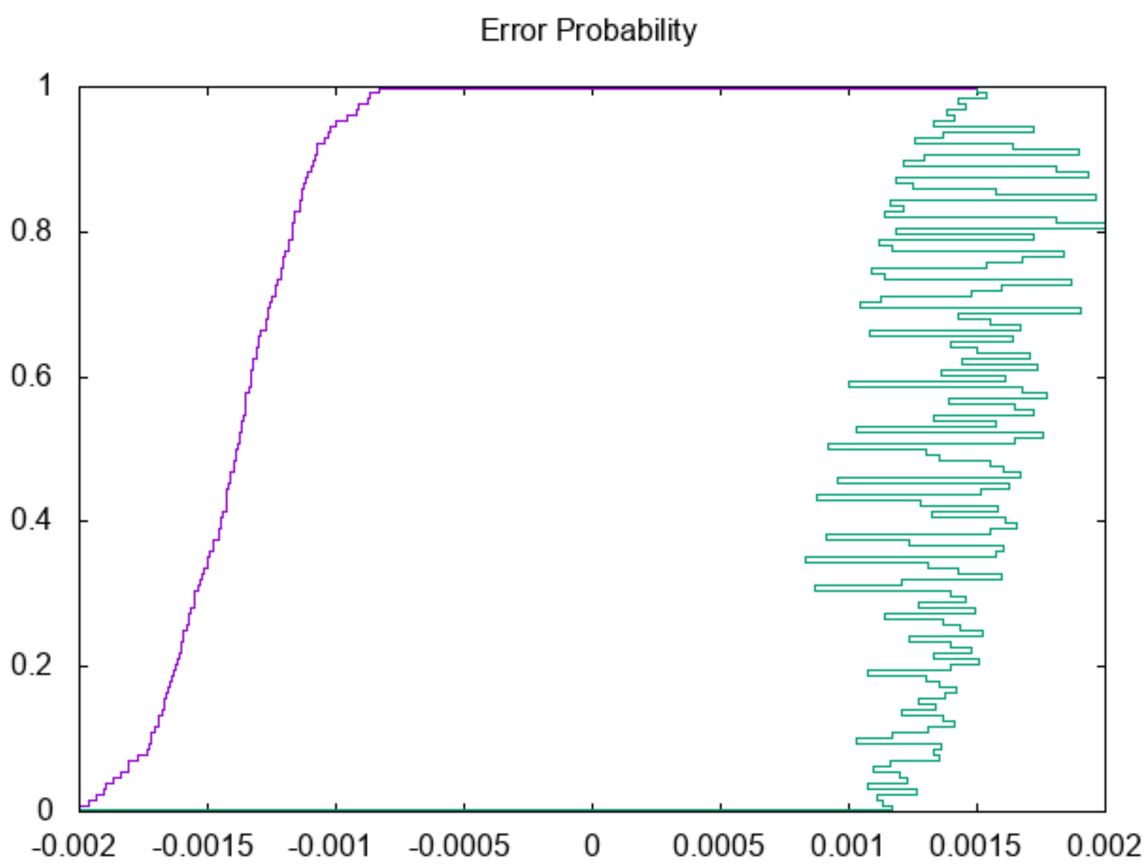
Error Metric Extraction

Threshold Probability

Error, Probability

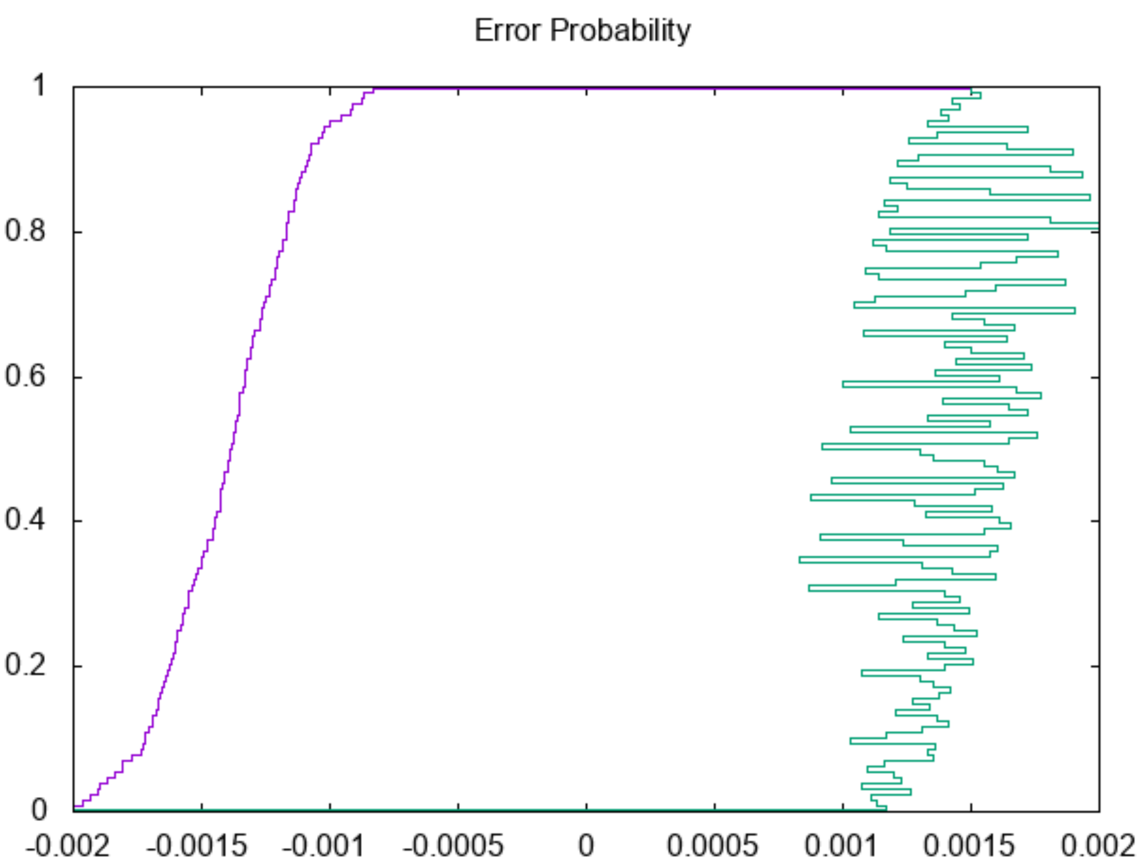
Computed Error Distribution

```
def func(..) {
 x := gaussian(0.0, 4.6)
 y := gaussian(0.0, 10.0)
 z := gaussian(0.0, 10.0)
 res = -3.79*x - 5.44*y + 9.73*z + 4.52
 return res
}
```



Error Metric Extraction

```
def func(..) {
 x := gaussian(0.0, 4.6)
 y := gaussian(0.0, 10.0)
 z := gaussian(0.0, 10.0)
 res = -3.79*x - 5.44*y + 9.73*z + 4.52
 return res
}
```



Error, Probability: **0.00176, 0.86**

Results: Prob. Error Analysis + Prob. Subdivision

Benchmarks	Worst case (state-of-the-art)	Prob. Subdivision (% Reduction)	Prob. Subdiv + Prob. Error (% Reduction)
sineOrder3	4.62E-07	-35.7	**-42.2**
sqrt	1.50E-04	**-44.1**	**-40.7**
bspline1	2.09E-07	**-6.2**	**-5.7**
rigidbody2	1.94E-02	-45.4	**-56.2**
traincar2	1.37E-03	-3.6	**-13.1**
filter4	6.51E-06	-6.5	**-23.8**
cubic	1.83E-05	**-5.5**	**3.8**
classIDX0	8.77E-06	-9.4	**-9.7**
polyIDX1	6.81E-04	**-33.8**	**-3.4**
neuron	3.22E-05	**-0.6**	**63.0**

Reduction % with 0.85 threshold probability for 32 bit floating-point and gaussian input distributions

Comparison

Benchmarks	Worst case (state-of-the-art)	Prob. Subdivision (% Reduction)	Prob. Subdiv + Prob. Error (% Reduction)
sineOrder3	4.62E-07	-35.7	**-42.2**
sqrt	1.50E-04	**-44.1**	-40.7
bspline1	2.09E-07	**-6.2**	-5.7
rigidbody2	1.94E-02	-45.4	**-56.2**
traincar2	1.37E-03	-3.6	**-13.1**
filter4	6.51E-06	-6.5	**-23.8**
cubic	1.83E-05	**-5.5**	**3.8**
classIDX0	8.77E-06	-9.4	**-9.7**
polyIDX1	6.81E-04	**-33.8**	-3.4
neuron	3.22E-05	**-0.6**	**63.0**

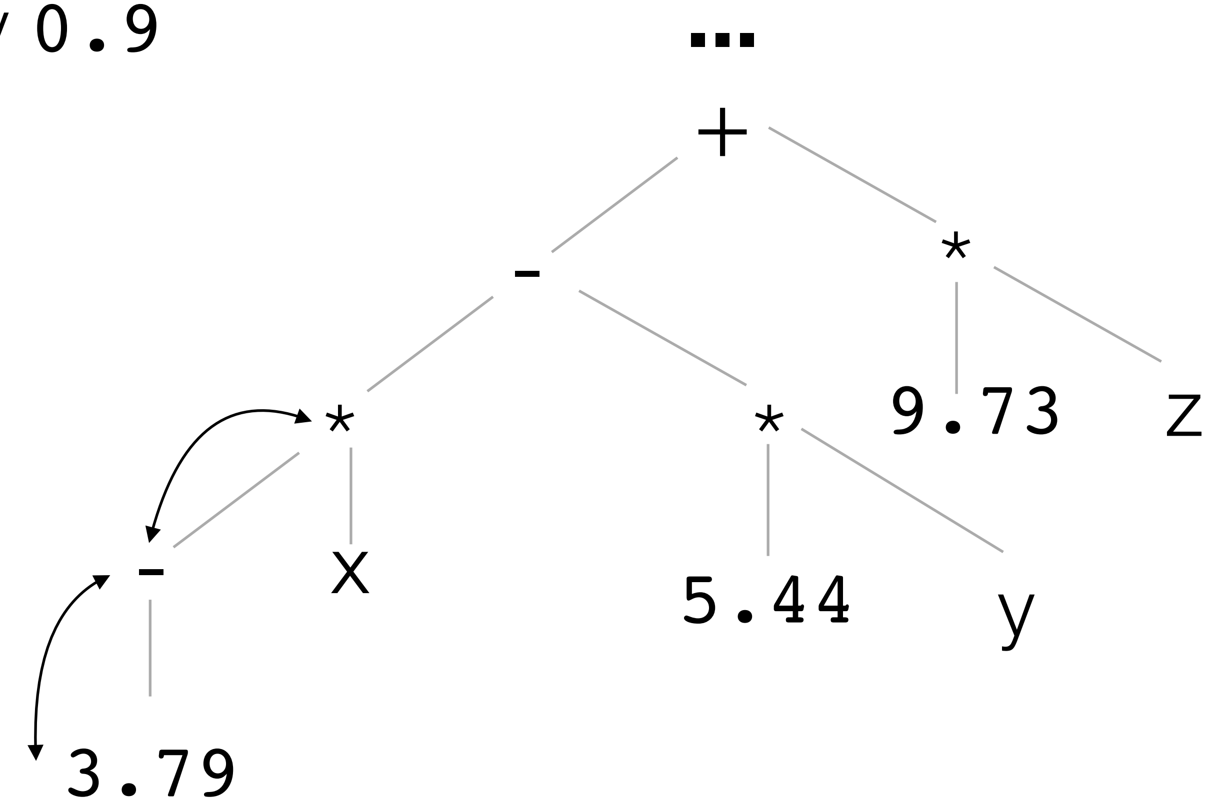
Reduction % with 0.85 threshold probability for 32 bit floating-point and gaussian input distributions

Performance depends on the application

What if we have **Approximate Hardware** with **Probabilistic Error Specifications?**

Probabilistic Error Specification

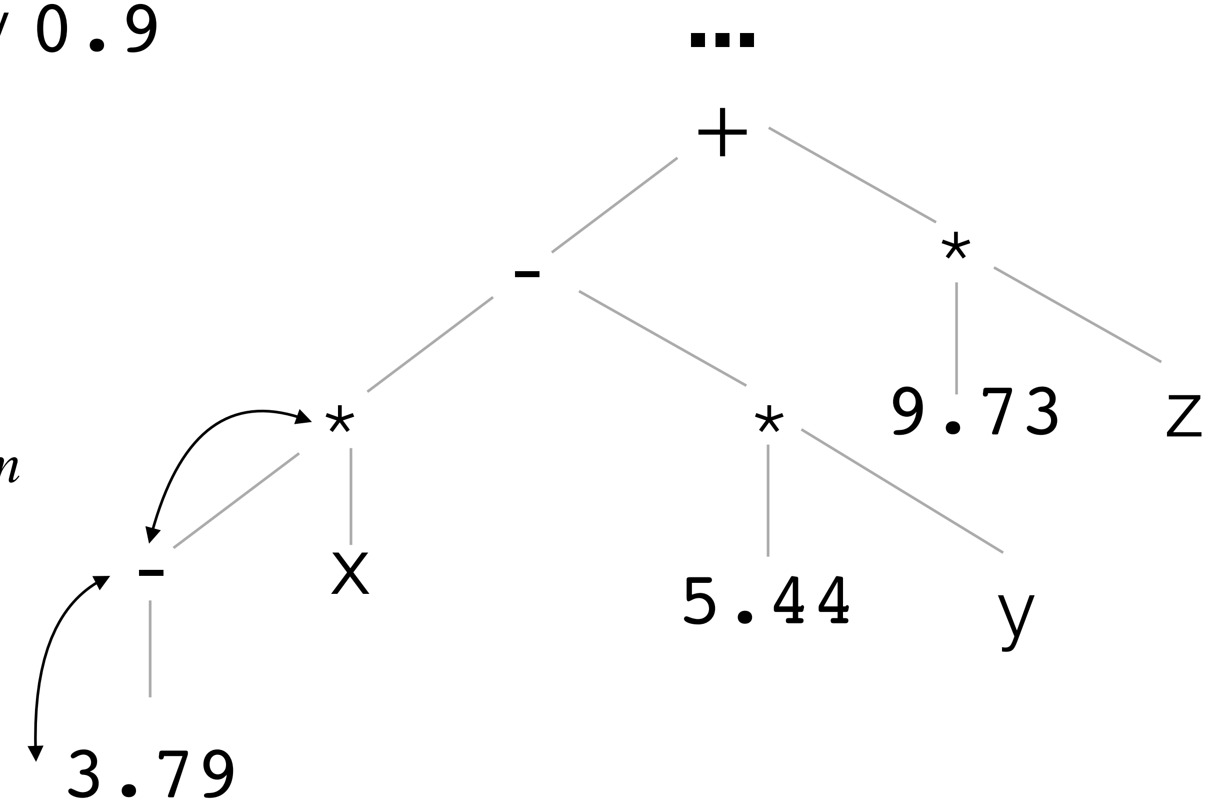
- Error: $4 \times \epsilon_m$, probability: `0.1`
- Error: $1 \times \epsilon_m$ with probability `0.9`



Worst Case Error Analysis

- Error: $4 \times \epsilon_m$, probability: `0.1`
- Error: $1 \times \epsilon_m$ with probability `0.9`

Worst-case: `3.79 +/-` $4 \times \epsilon_m$



- Worst Case Error Analysis can't utilize the probabilistic specification
- Not resource efficient

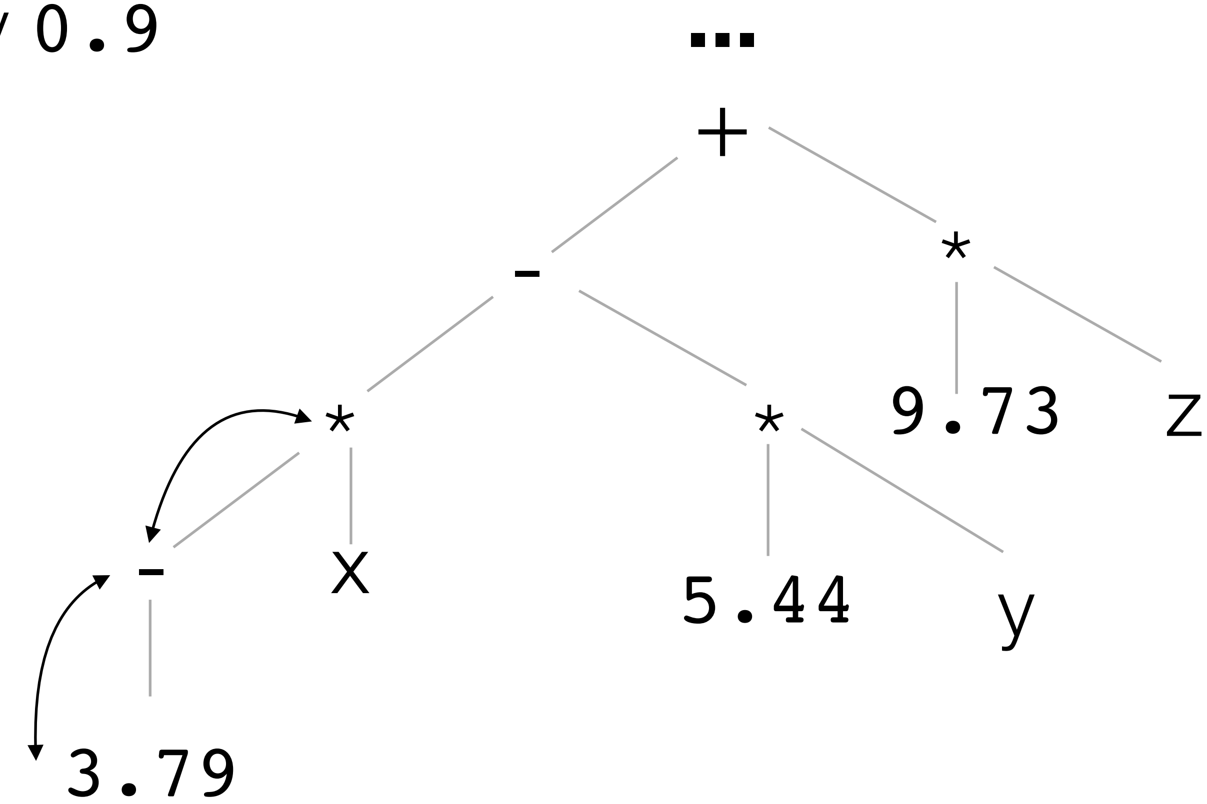
Probabilistic Error Analysis

- Error: $4 \times \epsilon_m$, probability: `0.1`
- Error: $1 \times \epsilon_m$ with probability `0.9`

Probabilistic error:

<[3.79 +/- ϵ_m], 0.9>,

<[3.79 +/- $4 \times \epsilon_m$], 0.1>



- Utilizes probabilistic error spec with probabilistic analysis
- Compute multiple errors for each operation

Results: Probabilistic Error Specification

Benchmarks	Prob Analysis + Prob Subdiv (100 subdivisions)
sineOrder3	-52.9%
sqrt	-56.6%
bspline1	-40.2%
rigidbody2	-13.5%
traincar2	-13.6%
filter4	-47.5%
cubic	-41.9%
classIDX0	-18.7%
polyIDX1	-10.6%
neuron	-41.7%

Reduction % with 0.85 threshold probability for 32 bit floating-point errors, gaussian input distributions considering $4 \times \epsilon_m$ error happens with 0.1 probability

Results: Probabilistic Error Specification

Benchmarks	Prob Analysis + Prob Subdiv (100 subdivisions)	Worst Case + Prob Subdiv (200 subdivisions)
sineOrder3	**-52.9%**	-34.0%
sqrt	**-56.6%**	-45.8%
bspline1	**-40.2%**	-9.7%
rigidbody2	-13.5%	**-49.2%**
traincar2	**-13.6%**	-1.9%
filter4	**-47.5%**	-10.4%
cubic	**-41.9%**	-9.3%
classIDX0	**-18.7%**	-13.6%
polyIDX1	-10.6%	**-37.3%**
neuron	**-41.7%**	-13.9%

Reduction % with 0.85 threshold probability for 32 bit floating-point errors, gaussian input distributions considering $4 \times \epsilon_m$ error happens with 0.1 probability

More in the paper

- Technical details of the probabilistic method
- Alternative approach to compute the error metric
- Case studies from embedded systems and machine learning
- More experiments with
  - uniform distribution of inputs
  - different error specifications

"Sound Probabilistic Numerical Error Analysis"

D. Lohar, M. Prokop, and E. Darulova

https://github.com/malyzajko/daisy/tree/probabilistic

Conclusion

- The first **Sound Analysis** of **Probabilistic Errors**
- **Interpretation** of the **error distribution** usable in real world
- Usage in applications with **Probabilistic Error Specification**